On commutators acting on some Fréchet spaces over non-Archimedean fields

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38th Summer Conference on Topology and its Applications
8-12 July 2024, Coimbra

Preliminaries

Let $\mathbb{K}$ be a field. A map $|\cdot|:\mathbb{K}\to[0,\infty)$ is a *valuation* if for all $\lambda,\mu\in\mathbb{K}$ we have the following:
(1) $|\lambda|=0$ if and only if $\lambda=0$,
(2) $|\lambda\mu|=|\lambda||\mu|$,
(3) $|\lambda+\mu|\leq|\lambda|+|\mu|$.

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Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \to [0, \infty)$ is a *valuation* if for all $\lambda, \mu \in \mathbb{K}$ we have the following:
(1) $|\lambda| = 0$ if and only if $\lambda = 0$,
(2) $|\lambda\mu| = |\lambda||\mu|$,
(3) $|\lambda + \mu| \leq |\lambda| + |\mu|$.

If $(\mathbb{K}, |\cdot|)$ is a valued field then there are only two possibilities. Either
- $\mathbb{K}$ is isomorphic to a subfield of $\mathbb{C}$ and $|\cdot|$ is equivalent with the ordinary absolute value function or
- the valuation $|\cdot|$ is *non-Archimedean* i.e. it satisfies *the strong triangle inequality*

$$|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$$

for all $\lambda, \mu \in \mathbb{K}$.

Preliminaries

Example: $(\mathbb{Q}_p, |\cdot|_p)$ - the field of p-adic numbers, where p is a prime number. The p-adic valuation $|\cdot|_p$ is determined by

$$|n/m|_p := p^{s-k}, n, m \in \mathbb{N},$$

where k and s are number of factors p in n and m, respectively.
The field $(\mathbb{Q}_p, |\cdot|_p)$ is locally compact.

We will always assume that $|\cdot|$ is *non-trivial* i.e., that there is a $\lambda_0 \in \mathbb{K}$ such that $|\lambda_0| \notin \{0, 1\}$.

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All linear spaces are over a non-archimedean non-trivially valued field $(\mathbb{K}, |\cdot|)$ which is complete under the metric induced by the valuation $|\cdot|$.

By a *seminorm* on a linear space E we mean a function
$p : E \to [0, \infty)$ such that
$p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and
$p(x + y) \leq max\{p(x), p(y)\}$ for all $x, y \in E$.

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A *Fréchet space* is a complete metrizable locally convex space.

A series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space E is convergent in E if and only if $\lim x_n = 0$.

By an operator on a lcs E we mean any continuous linear map on E. For any operator T on a Fréchet space E with a Schauder basis (e_n) there exists an infinite matrix $(t_{i,j})_{i,j\in\mathbb{N}}$ of scalars from $\mathbb{K}$ such that $Te_j = \sum_{i=1}^{\infty} t_{i,j} e_i$ for any $j \in \mathbb{N}$. Then

$$Tx = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} t_{i,j} x_j \right) e_i \text{ for any } x = \sum_{j=1}^{\infty} x_j e_j \in E.$$

Preliminaries

Let $B_{\mathbb{K}}$ be the closed unit ball of $\mathbb{K}$ (i.e.
$B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1.)$
A subset A of a lcs E is *absolutely convex* if $\alpha x + \beta y \in A$ for all $\alpha, \beta \in B_{\mathbb{K}}$ and $x, y \in A$.

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A subset B of a lcs E is *compactoid* (or a *compactoid*) if for each neighbourhood U of 0 in E there exists a finite subset A of E such that $B \subset U + \text{co}A$, where *coA* is the absolutely convex hull of A i.e.
$\text{co}A = \{\sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\}$

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If $\mathbb{K}$ is locally compact, then a subset B of a lcs E is compactoid if and only if it is precompact.

Preliminaries

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A lcs E is *nuclear* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $q \geq p$ such that the map

$$\varphi_{p,q} : (E_q, \overline{q}) \to (E_p, \overline{p})\,, x + \ker q \to x + \ker p$$

is compactoid.

A *commutator* of a pair of elements S and T in the algebra $L(E)$ of all operators on a lcs E is given by

$$[S, T] := ST - TS.$$

An infinite matrix $A = (a_{n,k})$ of real numbers is a *Köthe matrix* if $0 < a_{n,k} \leq a_{n,k+1}$ for all $n, k \in \mathbb{N}$.

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Let A be a Köthe matrix. The space
$K(A) = \{(x_n) \subset \mathbb{K} : \lim_{n\to\infty} |x_n| a_{n,k} = 0 \text{ for every } k \in \mathbb{N}\}$
with the base (p_k) of norms, where

$$p_k((x_n)) = \max_n |x_n| a_{n,k}, \, k \in \mathbb{N},$$

is a Fréchet space; it is called a Köthe space.
The sequence (e_j), where $e_j = (\delta_{j,n})$, is an unconditional Schauder basis of $K(A)$.

A Köthe space $K(A)$ is nuclear if and only if for any $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\lim_n \frac{a_{n,i}}{a_{n,j}} = 0.$$

Power series spaces

Let Γ be the family of all non-decreasing sequences $a = (a_n)$ of positive real numbers with $\lim_n a_n = \infty$. Let $a = (a_n) \in \Gamma$.

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Then the following Köthe spaces are the *power series spaces*:

1. $A_1(a) = K(B)$ with $B = (b_{n,k})$, $b_{n,k} = \exp(-a_n/k)$;
2. $A_\infty(a) = K(B)$ with $B = (b_{n,k})$, $b_{n,k} = \exp(ka_n)$.

(of *finite type* and *infinite type*, respectively).

For $r \in (-\infty, \infty]$ we denote by Λ_r the family of all strictly increasing sequences $(r_k) \subset \mathbb{R}$ with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$

Generalized power series spaces

For $r \in (-\infty, \infty]$ we denote by Λ_r the family of all strictly increasing sequences $(r_k) \subset \mathbb{R}$ with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$
and by Φ_c the family of all strictly increasing odd functions $f : \mathbb{R} \to \mathbb{R}$ such that f is convex in $[0, \infty)$.

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Let $f \in \Phi_c, a = (a_n) \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$.

Generalized power series spaces

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and by Φ_c the family of all strictly increasing odd functions $f : \mathbb{R} \to \mathbb{R}$ such that f is convex in $[0, \infty)$.

Let $f \in \Phi_c, a = (a_n) \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$.

Then the following Köthe space is said to be a *generalized power series space*

$$D_f(a, r) = K(B) \text{ with } B = (b_{n,k}), b_{n,k} = \exp(f(r_k a_n)).$$

Lemma A

Let $f \in \Phi_c$.

$$(1)\ f(u_1 v_2) - f(u_2 v_2) \le f(u_1 v_1) - f(u_2 v_1)$$

for $u_1, u_2, v_1, v_2 \in \mathbb{R}$ with $0 \le v_1 \le v_2$ and $u_1 \le u_2 \le 0$ or $0 \le u_1 \le u_2$;

$$(2)\ f(x) - f(y) \le f(x - y)$$

for all $x, y \in \mathbb{R}$ with $x \le y$ and $xy \ge 0$;

$$(3)\ f(x) - f(y) \le f(|x| + |y|) \text{ for all } x, y \in \mathbb{R}.$$

Results

Proposition 1

Let $D_f(a, r)$ be a generalized power series space.
(I) The left shift map on $D_f(a, r)$

$$L : D_f(a, r) \to D_f(a, r), (x_1, x_2, x_3, \dots) \to (x_2, x_3, x_4, \dots)$$

is well defined, linear and continuous if and only if
(1) $r > 0$ *or*
(2) $r = 0$ *and* $\sup_n [a_{n+1}/a_n] < \infty$ *or*
(3) $r < 0$ *and* $\lim_n [a_{n+1}/a_n] = 1$.
(II) The right shift map on $D_f(a, r)$

$$R : D_f(a, r) \to D_f(a, r), (x_1, x_2, x_3, \dots) \to (0, x_1, x_2, \dots)$$

is well defined, linear and continuous if and only if
(1) $r = \infty$ *and* $\sup_n [a_{n+1}/a_n] < \infty$ *or*
(2) $0 < r < \infty$ *and* $\lim_n [a_{n+1}/a_n] = 1$ *or (3)* $r \leq 0$.

Theorem 2

Let $D_f(a, r)$ be a generalized power series space. Assume that

$$(*)\ \forall k \in \mathbb{N}\, \exists p \in \mathbb{N}\, \forall s \in \mathbb{N}\, \exists t \in \mathbb{N} :$$

$$\sup_{n,m,v\in\mathbb{N}} [f(r_k a_{n+v}) - f(r_p a_v) + f(r_s a_m) - f(r_t a_{m+n})] < \infty.$$

Then every operator on $D_f(a, r)$ is a commutator.

Proof.

Theorem 3

Let $D_f(a, r)$ be a generalized power series space. Assume that $\sup_n [a_{2n}/a_n] < \infty$ and $r \in \{0, \infty\}$. Then every operator on $D_f(a, r)$ is a commutator.

Proof.

Theorem 3

Let $D_f(a, r)$ be a generalized power series space.
Assume that $\sup_n[a_{2n}/a_n] < \infty$ *and* $r \in \{0, \infty\}$.
Then every operator on $D_f(a, r)$ is a commutator.

Proof.

Corollary 4

Let $A_r(a)$ be a power series space.
If $\sup_n[a_{2n}/a_n] < \infty$, *then every operator on $A_r(a)$ is a commutator.*

A function $f \in \Phi_c$ is said to be *rapidly increasing if*

$$\lim_{t\to\infty}[f(ct)/f(t)] = \infty \text{ for any } c > 1.$$

Theorem 5

Let $D_f(a,r)$ be a generalized power series space. Assume that f is rapidly increasing, $\lim_n[a_{2n}/a_n] = 1$ and $r \in \mathbb{R}$ with $r \neq 0$. Then every operator on $D_f(a,r)$ is a commutator.

Example

Let $b \in \Gamma$ with $\lim_n[b_{n+1}/b_n] = 1$ (e.g. $b_n = n^c$ for $n \in \mathbb{N}, c > 0$). Put $a_n = b_k$ for $2^{k-1} \leq n < 2^k, k \in \mathbb{N}$. Then $a = (a_n) \in \Gamma$ and $\lim_n[a_{2n}/a_n] = 1$.

Proposition 6

Let $D_f(a, r)$ be a generalized power series space. Assume that (1) $\sup_n[a_{n+1}/a_n] < \infty$ *and* $r \in \{0, \infty\}$ *or* (2) $\lim_n[a_{n+1}/a_n] = 1$ *and* $r \in (\mathbb{R} \setminus \{0\})$. *Then every diagonal operator on $D_f(a, r)$ is a commutator; in particular, the identity operator I on $D_f(a, r)$ is a commutator.*

Theorem 7

Let $D_f(a, r)$ *be a generalized power series space with* $\lim_n[a_{n+1}/a_n] = \infty$*. Assume that an operator* T *on* $D_f(a, r)$ *is a commutator. Then* T *is bounded.*

Corollary 8

Let $D_f(a, r)$ *be a generalized power series space with* $\lim_n[a_{n+1}/a_n] = \infty$*. Then for any non-zero scalar* α *and any bounded operator* T *on* $D_f(a, r)$*, the operator* $\alpha I + T$ *is not a commutator. In particular, the identity operator* I *on* $D_f(a, r)$ *is not a commutator.*

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Thank you for your attention.