Raney extensions as pointfree T_0 spaces

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Structure of the talk

1. Saturated sets, pointfreely
2. Raney extensions
3. An application: exactness

The classical duality

The classical pointfree versions of topological spaces are *frames*. A frame is a complete lattice L where the distributivity law

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y)$$

holds. Frames form the category **Frm** when equipped with maps which preserve arbitrary joins and finite meets.

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holds. Frames form the category **Frm** when equipped with maps which preserve arbitrary joins and finite meets. We have an adjunction

$$\begin{tikzcd} \mathbf{Frm}^{op} \arrow[r, shift left=2, "\mathrm{pt}"] \arrow[r, phantom, "\top"] & \mathbf{Top} \arrow[l, shift left=2, "\Omega"] \end{tikzcd}$$

The fixpoints are the *sober* spaces on one side, and the *spatial* frames on the other. Sobriety is an axiom stronger than T_0 and weaker than T_2.

The T_D approach

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- A space X is sober if and only if there can be no nontrivial subspace inclusion $i : X \subseteq Y$ such that $\Omega(i)$ is an isomorphism;
- A space X is T_D if and only if there can be no nontrivial subspace inclusion $i : Y \subseteq X$ such that $\Omega(i)$ is an isomorphism.

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In [Banaschewski and Pultr, 2010], an alternative approach to the classical one is introduced, where to every frame L a T_D spectrum $\mathrm{pt}_D(L)$ is associated. The points are the *covered* primes, i. e. those such that $\bigwedge_i x_i = p$ implies $x_i = p$ for some i.

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Theorem

There is an adjunction $\mathrm{pt}_D : \mathbf{Frm}_D^{op} \leftrightarrows \mathbf{Top}$, *whose fixpoints are the D-spatial frames on one side and the* T_D *spaces on the other.*

Raney duality

Raney duality (see [Bezhanishvili and Harding, 2020]) enables us to faithfully represent all the T_0 spaces. The core intuition is that a T_0 space can be completely recovered from the embedding $\Omega(X) \subseteq \mathcal{U}(X)$, as the space of completely join-prime elements.

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- D is a completely distributive lattice *generated by its completely join-prime elements*;
- $L \subseteq D$ is a frame embedding;
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Morphisms of Raney algebras are complete lattice morphisms that respect the designated frame. We call the category **RA**. For any space X, the pair $(\Omega(X), \mathcal{U}(X))$ is a Raney algebra. For a Raney algebra (L, D), one can define the topological space $\mathrm{pt}_{RA}(L, D)$ of completely join-prime elements of D.

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Theorem

There is a dual adjunction $\Omega_{RA} : \mathbf{Top} \leftrightarrows \mathbf{RA}^{op} : \mathrm{pt}_{RA}$. The fixpoints are T_0 spaces on one side, and all Raney algebras on the other.

Raney duality beyond points?

Raney duality has an issue: all Raney algebras are spatial. The obvious forgetful functor $\mathbf{RA} \to \mathbf{Frm}$ only reaches the spatial frames; in other words: only spatial frames can be extended to Raney algebras. But we would like to be able to extend any frame in such fashion.

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The pair $(\Omega_{RA}, \mathrm{pt}_{RA})$ being an equivalence (in particular, essential surjectiveness of pt_{RA}) relies on there being enough completely join-primes in a Raney algebra. To do away with points, we plan to use the following observation.

Observation

Even if we drop the constraint that in (L, D) the lattice D is generated by the completely join-prime elements, the pair of functors $(\Omega_{RA}, \mathrm{pt}_{RA})$, suitably extended, is still well-defined, and a dual adjunction...

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Raney extensions

We define a *Raney extension* as a pair (L, C) such that:

- C is a coframe;
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Morphisms are defined as coframe maps that respect the designated frame. We call the category **Raney**. The original dual equivalence by Raney can easily be extended to the following.

Theorem

There is a dual adjunction $\Omega_R : \mathbf{Top} \leftrightarrows \mathbf{Raney}^{op} : \mathrm{pt}_R$. The fixpoints are T_0 spaces on one side, and Raney algebras on the other.

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- The pair (L, L^{δ}) for any pre-spatial frame L, where L^{δ} is the canonical extension of L as defined in [Jakl, 2020],
- The pair $(L, \overline{B(L)})$ for a subfit frame L, where $\overline{B(-)}$ is the Funayama construction (see, for example, [Bezhanishvili et al., 2013]).

Raney extensions

Every frame has the largest and the smallest Raney extension. In [Suarez, 2024a], the following is shown.

Theorem

For any Raney extensions (L, C), we have surjections

$$(L, \mathrm{Filt}_{\mathcal{SE}}(L)^{op})) \twoheadrightarrow (L, C) \twoheadrightarrow (L, \mathrm{Filt}_{\mathcal{E}}(L)^{op}).$$

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This result shows that the structures $\mathsf{S}_{\mathfrak{o}}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)^{op}$, well-studied in pointfree topology, enjoy universal properties which are dual of one another.

Raney extensions

What happens if we take the spectrum pt_R of the two embedding in the theorem above?

Theorem

For any Raney extensions (L, C), we have subspace inclusions

$$\mathrm{pt}_D(L) \subseteq \mathrm{pt}_R(L, C) \subseteq \mathrm{pt}(L),$$

where $\mathrm{pt}(L)$ is the classical spectrum of L, and $\mathrm{pt}_D(L)$ its T_D spectrum, as introduced in [Banaschewski and Pultr, 2010].

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This gives another sense in which the T_D axiom is a mirror image of sobriety, apart from that described in [Banaschewski and Pultr, 2010].

Lifting of morphisms

Every Raney extension may be seen as a collection of filters of the base frame. In fact, we have an adjunction

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\begin{tikzcd}
C \arrow[r, shift left=2, "\uparrow^L"] \arrow[r, phantom, "\perp"] & \mathrm{Filt}(L)^{op} \arrow[l, shift left=2, "\bigwedge"]
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where $\uparrow^L c = \{a \in L : x \leq a\}$. This restricts to an isomorphism between C and some subcolocale of $\mathrm{Filt}(L)^{op}$.

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Theorem

A morphism $f : L \to M$ extends to a Raney morphism $(L, \mathcal{F}) \to (M, \mathcal{G})$ if and only if $f^{-1}(G) \in \mathcal{F}$ for all $G \in \mathcal{G}$.

Exact morphisms

We now show some results from [Suarez, 2024b] showing applications to exactness and the T_D duality. From the result we know that the assignment $L \mapsto (L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ is functorial precisely for those morphisms such that preimages of exact filters are exact.

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Proposition

A morphism $f : L \to M$ of frames extend to a frame morphism $\mathsf{S}_{\mathfrak{c}}(L) \to \mathsf{S}_{\mathfrak{c}}(M)$ if and only if it is exact.

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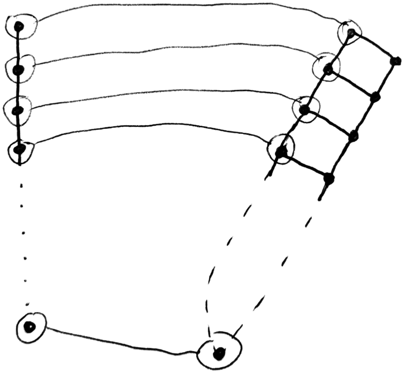
For a frame L, the following are all exact maps.

- The surjection corresponding to an open sublocale.
- The surjection corresponding to a closed sublocale.
- The surjection corresponding to a d-point.

Exact morphisms: counterexamples

- $f(\bigwedge_i x_i) = \bigwedge_i f(x_i)$
- $\bigwedge_i f(x_i)$ not exact



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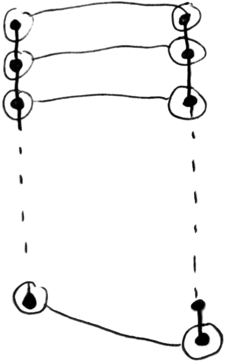

Exact sublocales

For a frame L we call a sublocale *exact* if the corresponding surjection is exact.

Theorem

We have a chain of subcolocale inclusions

$$\mathsf{S}_{\mathcal{E}}(L) \subseteq \mathsf{S}_D(L) \subseteq \mathsf{S}(L),$$

where $\mathsf{S}_D(L)$ is the collection of D-sublocales in [Arrieta and Suarez, 2021].

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Not all d-morphisms are exact. See the first counterexample before.

Exactness and the T_D duality

Let us now look at the connection with T_D duality. Restricting $\mathbf{Frm}_D$ further, to $\mathbf{Frm}_{\mathcal{E}}$,does not disrupt T_D duality, while giving us functoriality of $L \mapsto \mathsf{S}_{\mathfrak{c}}(L)$.

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Lemma

If $f : L \to M$ is a d-morphism and M is T_D-spatial, then f is exact.

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Lemma

If $f : L \to M$ is a d-morphism and M is T_D-spatial, then f is exact.

Theorem

We have a commuting diagram as follows, where $\mathsf{S}_{\mathfrak{c}} : L \mapsto (L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$.

$$\begin{tikzcd}
{\mathbf{Frm}_{\mathcal{E}}}^{op} \arrow[r, "\mathsf{S}_{\mathfrak{c}}"] \arrow[dr, "\mathrm{pt}_D"'] & \mathbf{Raney}^{op} \arrow[d, "\mathrm{pt}_R"] \\
 & \mathbf{Top}
\end{tikzcd}$$

Raney as bringing together the sober and T_D dualities

In fact, the Raney approach encompasses both the classical and the T_D approaches to pointfree topology. The following commutes.

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\begin{tikzcd}
\mathbf{Frm}_{\mathcal{E}}{}^{op} \arrow[r, "\mathsf{S}_{\mathfrak{c}}"] \arrow[dr, "\mathrm{pt}_D"'] & \mathbf{Raney}^{op} \arrow[d, "\mathrm{pt}_R"] & \mathbf{Frm}^{op} \arrow[l, "\mathsf{S}_{\mathfrak{o}}"'] \arrow[dl, "\mathrm{pt}"] \\
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& \mathbf{Top} &
\end{tikzcd}$$

In other words, the T_D duality and the sober duality are both restrictions of Raney duality: each frame is identified, respectively, with the smallest and the largest Raney extension on it. In this sense, the sober duality and the T_D duality are mirror images of one another.

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