Orbit structure in CR-dynamical systems

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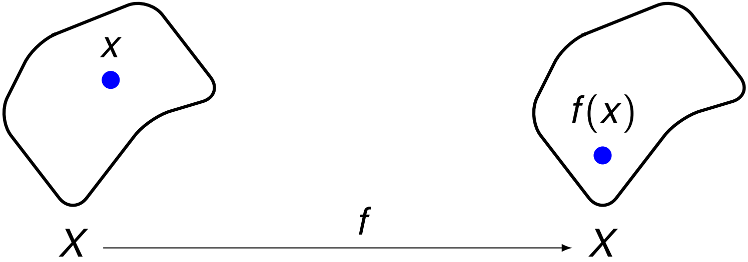
Topological Dynamical Systems

Definition

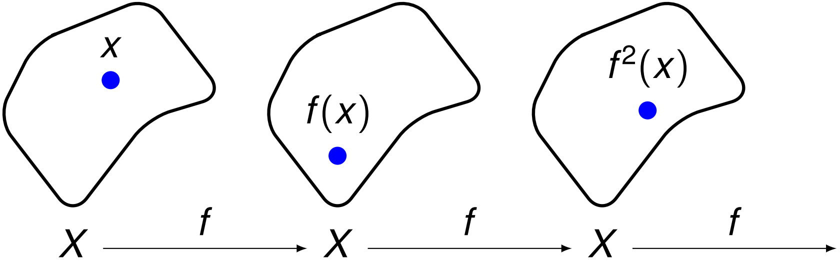
For a non-empty compact metric space X and continuous function $f : X \to X$, we say (X, f) is a *topological dynamical system*.



Topological Dynamical Systems

Definition

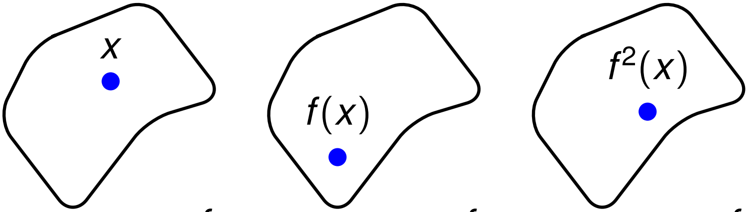
Let (X, f) be a topological dynamical system. Then the *trajectory* of a point $x \in X$ is the sequence $\langle x, f(x), f^2(x), \ldots \rangle$.



Topological Dynamical Systems

Definition

Let (X, f) be a topological dynamical system. We say $x \in X$ is a *transitive point*, if $\{x, f(x), f^2(x), \ldots\}$ is dense in X.

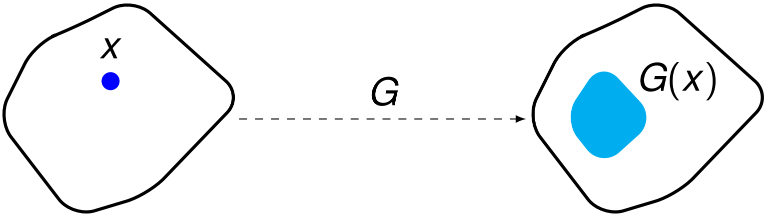


$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f}$$

CR-Dynamical Systems

Definition

For a non-empty compact metric space X and non-empty closed $G \subseteq X \times X$, we say (X, G) is a *CR-Dynamical System*. For each $x \in X$, we denote by $G(x)$, the set $\{y \in X \mid (x, y) \in G\}$.



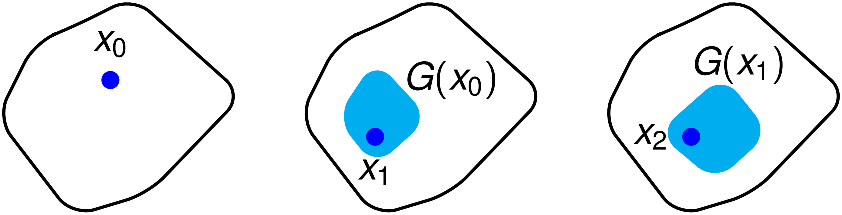
Trajectories

Definition

Let (X, G) be a CR-dynamical system. Then a *trajectory* of a point $x \in X$ is a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ such that

- $x_0 = x$; and
- $x_{n+1} \in G(x_n)$ for all $n \in \mathbb{N}$.

Denote by $T_G^+(x)$ the set of all trajectories of x.

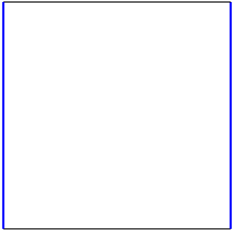


Legality

Definition

Suppose (X, G) is a CR-dynamical system.

- We say $x \in X$ is *legal* if $T_G^+(x) \neq \varnothing$
- We say $x \in X$ is *illegal* if $T_G^+(x) = \varnothing$.
- Denote by $\text{legal}(G)$ the set of legal points.



$X = [0, 1]$

$G = \{0, 1\} \times X$

$\text{legal}(G) = \{0, 1\}$

Orbit structure in classical dynamics

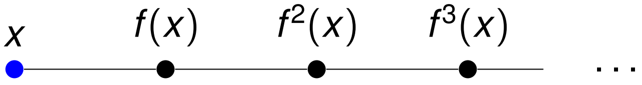


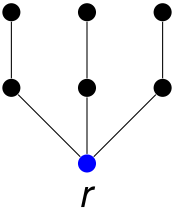
Figure: Orbit structure in topological dynamical systems

Trees

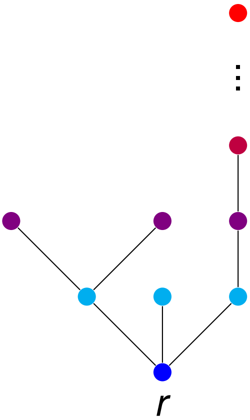
Definition

We say a partially ordered set $(T, \leq)$ is a *rooted tree*, if there exists a unique point $r \in T$, called the *root*, such that for each $x \in T$:

- $r \leq x$.
- $(\{y \in T \mid y \leq x\}, \leq)$ is well-ordered.



Levels



$\mathrm{level}_n(T)$

$\vdots$

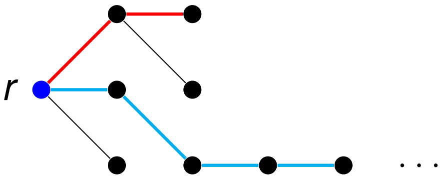
$\mathrm{level}_3(T)$

$$\left.\begin{matrix}\mathrm{level}_2(T)\\ \mathrm{level}_1(T)\\ \mathrm{level}_0(T)\end{matrix}\right]\; \mathcal{L}_2(T)$$

Branches

Definition

Let $(T, \leq)$ be a tree. We call a maximally well-ordered subset of T a *branch*. We denote by $\mathcal{B}_\infty(T)$, the set of infinite branches in T.



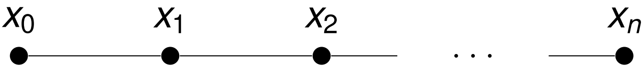
Paths in G

Definition

Let (X, G) be a CR-dynamical system, $x, y \in X$ and $n \in \mathbb{N}$. We say *$\gamma = x_0 \dots x_n$ is an x-path in G*, if

- $x_0 = x$; and
- $(x_k, x_{k+1}) \in G$ for each $k \in \{0, 1, \dots, n-1\}$.

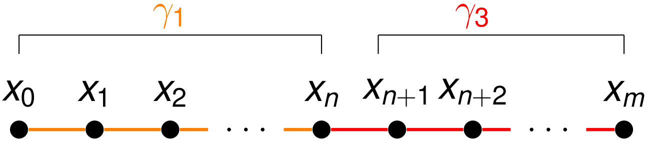
We denote by x_γ the endpoint of our path γ, i.e., $x_\gamma = x_n$. If in addition $x_\gamma = y$, we say *$\gamma = x_0 \dots x_n$ is a path from x to y in G*. We denote by $\Gamma_G(x)$, the set of x-paths in G.



Extension of paths in G

Definition

Let (X, G) be a CR-dynamical system and $x \in X$. Let $\gamma_1, \gamma_2 \in \Gamma_G(x)$. We say γ_1 *extends* to γ_2, if $\gamma_2 = \gamma_1\gamma_3$ for some path γ_3 in G.



Extension of paths in G

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Theorem

Let (X, G) be a CR-dynamical system and $x \in X$. Then, $(\Gamma_G(x), \leq)$ is a tree, where

$$\gamma_1 < \gamma_2 \text{ if, and only if, } \gamma_1 \text{ extends to } \gamma_2,$$

for each $\gamma_1, \gamma_2 \in \Gamma_G(x)$.

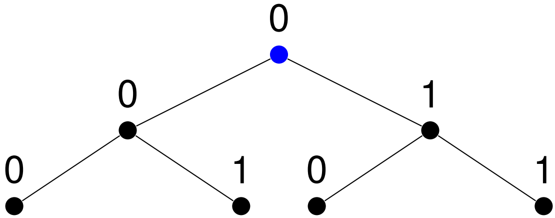
Transitivity trees

Definition

Let (X, G) be a CR-dynamical system. The *Transitivity tree of (X, G) with respect to x*, is the tree $(T_G(x), \leq)$, where $T_G(x) = \Gamma_G(x)$ and $\leq$ is the path extension order. If $S \subseteq T_G(x)$, we denote by S^* the set $\{x_\gamma \in X \mid \gamma \in S\}$.

Example

Let $X = \{0, 1\}$ and $G = X \times X$. Then, below we show $\mathcal{L}_2(T_G(0))$.



We note $T_G(0)$ consists of all binary strings starting at 0, and $T_G(1)$ all binary strings starting at 1.

Legal Points

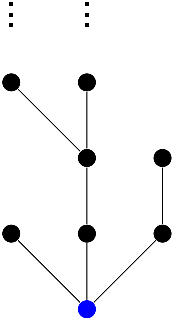


Figure: Transitivity tree of a legal point

Application to transitive points

Definition

Let (X, G) be a CR-dynamical system. We say $x \in \text{legal}\,(G)$ is

- 1*-transitive*, if for every infinite branch $B \in \mathcal{B}_\infty(T_G(x))$ we have $\overline{B^*} = X$.
- 2*-transitive*, if there exists an infinite branch $B \in \mathcal{B}_\infty(T_G(x))$ such that $\overline{B^*} = X$.
- 3*-transitive*, if $\bigcup_{B \in \mathcal{B}_\infty(T_G(x))} B^*$ is dense in X.

$$\text{trans}_1\,(G) \subseteq \text{trans}_2\,(G) \subseteq \text{trans}_3\,(G)$$

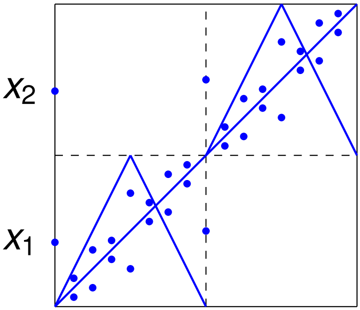
Application to transitive points

Theorem

Let (X, G) *be a CR-dynamical system such that* $\text{isolated}(X) \neq \varnothing$ *and* $\text{trans}_2(G) \neq \varnothing$. *Then, there exists* $x \in X$ *such that*

$$T_{G^{-1}}(x)^* = \text{trans}_2(G) = \text{trans}_3(G).$$

Stormy Campsite Map



Transitive points is G_δ

Theorem

Let (X, f) be a topological dynamical system, and $\mathscr{U}$ be a countable base for X. Then,

$$\mathrm{trans}\,(f) = \bigcap_{U \in \mathscr{U}} \left(\bigcup_{k=0}^{\infty} f^{-k}(U) \right).$$

Generalisation

Theorem

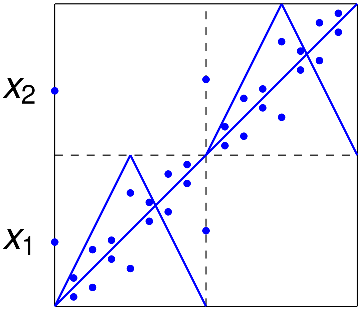
Let (X, G) be a CR-dynamical system, and $\mathscr{U}$ be a countable base for X. Then,

$$\operatorname{trans}_3(G) = \bigcap_{U \in \mathscr{U}} \left(\bigcup_{k=0}^{\infty} G^{-k}(U) \right),$$

where

$$G^{-k}(U) := \{x \in X \mid \operatorname{level}_k(T_G(x))^* \cap U \neq \varnothing\}.$$

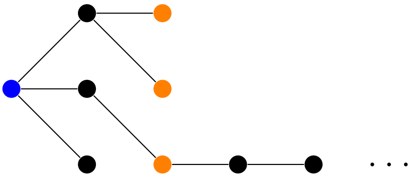
3-transitive points is not G_δ



0-transitive points

Definition

Let (X, G) be a CR-dynamical system. We say $x \in \text{legal}(G)$ is *0-transitive*, if for every non-empty open $U \subseteq X$, there exists $n \in \mathbb{N}$ such that $\text{level}_n(T_G(x))^* \subseteq U$.



Generalisation

Theorem

Let (X, G) be a CR-dynamical system, and $\mathscr{U}$ be a countable base for X. Then,

$$\mathrm{trans}_0\left(G\right) = \bigcap_{U \in \mathscr{U}} \left(\bigcup_{k=0}^{\infty} G^{-k}[U] \cap \mathrm{legal}\left(G\right) \right),$$

where

$$G^{-k}[U] := \{x \in X \mid \mathrm{level}_k\left(T_G(x)\right)^* \subseteq U\}.$$

Relationship to other transitive points

$$\mathrm{trans}_0(G) \subseteq \mathrm{trans}_1(G) \subseteq \mathrm{trans}_2(G) \subseteq \mathrm{trans}_3(G)$$

Existence

Question

Do any interesting 0-transitive points exist?

Example

Let $X = [0,1]$ and $C = \prod_{i=1}^{\infty} \{0,1\}$.

- Let $D : X \to 2^X$ be the doubling map, i.e.,

$$D(x) = \begin{cases} \{2x\} & \text{if } x \in [0, \frac{1}{2}); \\ \{0,1\} & \text{if } x = \frac{1}{2}; \\ \{2x-1\} & \text{if } x \in (\frac{1}{2}, 1]; \end{cases}$$

- $\sigma : C \to C$ denote the shift map, i.e.,

$$\sigma(c_1, c_2, c_3, \ldots) = (c_2, c_3, \ldots).$$

- $h : C \to X$ be defined by

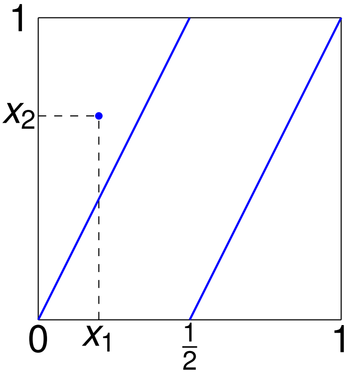
$$h(c_1, c_2, \ldots) = \sum_{i=1}^{\infty} \frac{c_i}{2^i}.$$

- Forms a semi-conjugacy: $h \circ \sigma = D \circ h$

Example

- $0, 1, \{0,0\}, \{0,1\}, \{1,0\}, \{1,1\}, \ldots$
- $s_0 = 0010000010100110\ldots$
- $s_1 = 0111001011101111\ldots$
- $x_1 = \frac{h(s_0)}{2}$ and $x_2 = h(s_1)$

Example



$$G = \text{Graph}\,(D) \cup \{(x_1, x_2)\}$$

Thank you for your attention!