Continuum-wise hyperbolicity

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Linear Anosov Diffeomorphisms

- $A \in SL(2, \mathbb{Z})$ hyperbolic

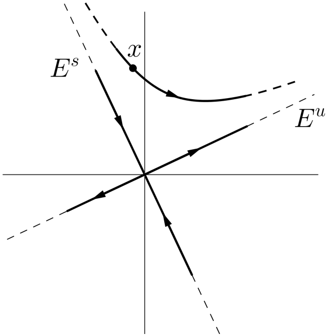

Figure: Stable/unstable spaces

Linear cw-Anosov

- A induces diffeomorphisms on the torus $\mathbb{T}^2$ and on the sphere $\mathbb{S}^2 = \mathbb{T}^2/\{x \sim -x\}$

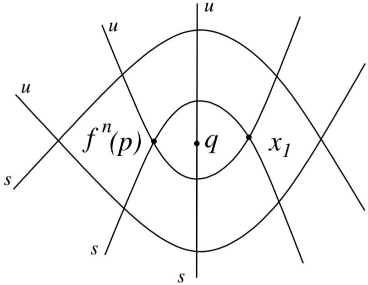

Figure: Singularity of the Pseudo-Anosov in $\mathbb{S}^2$

Local stable/unstable sets

- (X, d) compact metric space, $f : X \to X$ a homeomorphism, $x \in X$ and $\varepsilon > 0$
- $W^s_\varepsilon(x) = \{y \in X \;\; ; \;\; d(f^k(x), f^k(y)) \leq \varepsilon, \;\; \forall k \in \mathbb{N}\}$
- $W^u_\varepsilon(x) = \{y \in X \;\; ; \;\; d(f^{-k}(x), f^{-k}(y)) \leq \varepsilon, \;\; \forall k \in \mathbb{N}\}$
- $\Gamma_\varepsilon(x) = W^s_\varepsilon(x) \cap W^u_\varepsilon(x)$ (Bowen's dynamical ball)
- f is expansive if $\exists \; \varepsilon > 0$ such that $\Gamma_\varepsilon(x) = \{x\} \;\; \forall x \in X$

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Continuum-wise hyperbolicity

- Continuum-wise expansiveness: $\exists\ \varepsilon > 0$ such that $\Gamma_\varepsilon(x)$ is totally disconected $\ \forall x \in X$

- $C^s_\varepsilon(x)$ is the connected component of x in $W^s_\varepsilon(x)$

- $C^u_\varepsilon(x)$ is the connected component of x in $W^u_\varepsilon(x)$

- Continuum-wise local product structure: $\forall\ \varepsilon > 0,\ \exists\ \delta > 0;$

$$d(x, y) < \delta \ \text{ implies } \ C^u_\varepsilon(x) \cap C^s_\varepsilon(y) \neq \emptyset.$$

- Continuum-wise hyperbolicity: cw-expansiveness + cw-lps.

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Cw$_F$-hyperbolicity

- Cw$_F$-expansiveness: there exists $\varepsilon > 0$ such that $C^s_\varepsilon(x) \cap C^u_\varepsilon(x)$ is finite for every $x \in X$
- Cw$_F$-hyperbolicity: cw-hyperbolicity + cw$_F$-expansiveness
- Cw$_N$-expansiveness: $\#(C^s_\varepsilon(x) \cap C^u_\varepsilon(x)) \leq N$ for every $x \in X$
- Cw$_N$-hyperbolicity: cw-hyperbolicity + cw$_N$-expansiveness
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Local stable/unstable continua

- Cw_F-hyperbolicity on surfaces implies local stable/unstable continua are arcs
- (A. Artigue) Examples of cw-expansive homeomorphisms on surfaces where local stable/unstable continua are dendrites with a cantor set of spines and bifurcation points
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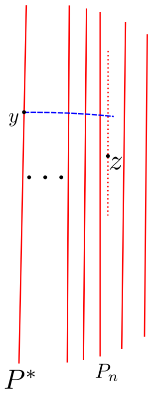
Local stable/unstable continua

Step 1: Local stable/unstable continua are locally connected



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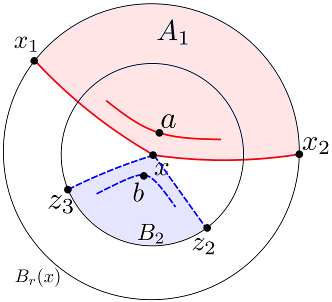
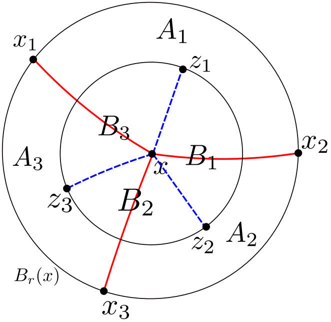
Step 2: Local stable/unstable arcs intercalate



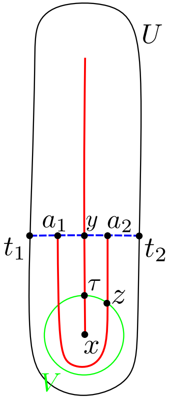
Local stable/unstable continua

Step 3: There are no branching points



Local stable/unstable continua

Spines are contained in regular bi-asymptotic sectors and are isolated from other spines



Local stable/unstable continua

Theorem 1 (Arruda, C., Sarmiento)

If a cw$_F$-hyperbolic surface homeomorphism has only a finite number of spines, then it is cw2-hyperbolic. Cw3-hyperbolic surface homeomorphisms have at most a finite number of spines, and are cw2-hyperbolic.

Ongoing: Cw$_F$-hyperbolic homeomorphisms on surfaces are either conjugate to a linear Anosov on $\mathbb{T}^2$ or a linear cw-Anosov on $\mathbb{S}^2$.

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Articles

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- R. Arruda, B. Carvalho, A. Sarmiento. Continuum-wise hyperbolic homeomorphisms on surfaces. *Discrete and Continuous Dynamical Systems* (2024) **44** (3) 768-790.
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