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SUMTOPO 2024

GENERICALLY HEREDITARILY EQUIVALENT PEANO CONTINUA

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joint work with Benjamin Vejnar (Charles University)

Introduction

Definition 1.1

A compact, connected, and metrizable space is a **continuum**. If it is locally connected, it is a **Peano continuum**.

Definition 1.2

A continuum X is **hereditarily equivalent** if every non-degenerate subcontinuum of it is homeomorphic to X. In this case we say X is HEC.

Definition 1.3

If X is a continuum, $\mathrm{Cont}(X)$ is the hyperspace of subcontinua of X. $\mathrm{Fin}_1(X)$ is the hyperspace of singletons of X.

Note that if X is HEC

$$\mathscr{G} = \{K \in \mathrm{Cont}(X) \mid K \simeq X\} = \mathrm{Cont}(X) \setminus \mathrm{Fin}_1(X)$$

so $\mathscr{G}$ is a comeager subset of $\mathrm{Cont}(X)$.

Definition 1.4

A continuum X is **generically hereditarily equivalent** if

$$\{K \in \mathrm{Cont}(X) \mid K \simeq X\}$$

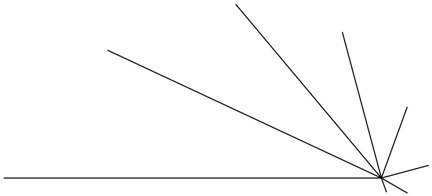
is comeager in $\mathrm{Cont}(X)$. In this case we say X is GHEC.

Generalized Ważewski dendrite is GHEC

Definition 2.1

A **dendrite** is a Peano continuum without homeomorphic copies of the circle.

The universal Ważewski dendrite is such that every dendrite can be embedded into it. How can one construct it?



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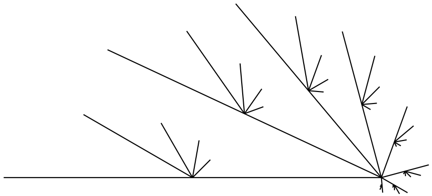

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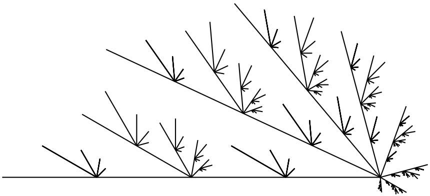

Definition 2.2

Given a topological space X and $A \subseteq X$, the **order** of A in X is the least cardinal number α for which every open set $U \supseteq A$ there exists open set V such that

$$A \subseteq V \subseteq U \text{ and } |\partial V| \leq \alpha.$$

We write that

$$\operatorname{ord}(A, X) = \alpha.$$

Definition 2.3

Let $M \subseteq \{3, 4, 5, \ldots\} \cup \{\omega\}$. The dendrite W_M is defined as the dendrite whose set of branching points are of order $m \in M$ and for all $m \in M$

$$\{x \in W_M \mid \operatorname{ord}(x, W_M) = m\}$$

is arcwise dense in W_M.

Definition 2.4

Let X, Y be dendrites with $X \subseteq Y$. The first point map $r_{Y,X} : Y \to X$ takes $y \in Y$ to the first point in the arc starting from y to any $x \in X$ that is also in X.

Lemma 2.5

Let X, Y be dendrites with $X \subseteq Y$. A branching point x of X is **maximal** in Y if one of the following are satisfied:

(i) $r_{Y,X}^{-1}(x)$ is degenerate ($|r_{Y,X}^{-1}(x)| = 1$).

(ii) there is no arc $A \subseteq Y$ from $y \in Y \setminus X$ to x with $A \cap X = \{x\}$.

(iii) every open neighborhood of b in X meets each component of $Y \setminus \{b\}$.

Theorem 2.6

Given $M \subseteq \{3, 4, \ldots\} \cup \{\omega\}$, $M \neq \emptyset$, the generic subcontinua of W_M is homeomorphic to W_M.

Proof's sketch:

- $\mathcal{B}(W_M)$ be the collection of branching points of W_M
- $\mathscr{M}(b, \ell) = \left\{ K \in \mathrm{Cont}(W_M) \;\middle|\; \begin{array}{c} b \in K \text{ and } K \text{ does not meet} \\ \text{the } \ell\text{-th component of } W_M \setminus \{b\} \end{array} \right\}$
- $\mathscr{M}(b, \ell)$ is closed: If $K \in \mathrm{Cont}(W_M) \setminus \mathscr{M}(b, \ell)$ either $b \notin K$ or $b \in K$ but K meets the ℓ-th component U of $W_M \setminus \{b\}$. Thus,

$$K \in \langle \{b\}^c \rangle \quad \text{or} \quad K \in \langle X, U \rangle .$$

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Given $M \subseteq \{3, 4, \ldots\} \cup \{\infty\}$, $M \neq \emptyset$, the generic subcontinua of W_M is homeomorphic to W_M.

Proof's sketch:

- $\mathscr{M}(b, \ell)$ has an empty interior: Given $K \in \mathscr{M}(b, \ell)$ and $\varepsilon > 0$, there exists a connected open neighborhood V of b contained in $B(b, \varepsilon/2)$. Hence, $K' = K \cup \overline{V} \in \mathrm{Cont}(W_M) \setminus \mathscr{M}(b, \ell)$ is at distance smaller than ε from K

$$\mathscr{G} = \mathrm{Cont}(W_M) \setminus \left(\left(\bigcup_{b \in \mathscr{B}(W_M)} \bigcup_{\ell} \mathscr{M}(b, \ell) \right) \cup \mathrm{Fin}_1(W_M) \right)$$

$$= \left\{ K \in \mathrm{Cont}(W_M) \;\middle|\; \begin{array}{c} K \text{ is non-degenerate and if } b \in \mathcal{B}(W_M) \cap K, \\ \text{then } b \in \mathcal{B}(K) \text{ and it is maximal in } W_M \end{array} \right\}$$

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Theorem 2.7

If X is a Peano GHEC, then X is an arc or a dendrite such that for every $m \in \{3, 4, \ldots\} \cup \{\omega\}$ the collection of branching points of order m is either dense or empty. Moreover, the collection of branching points is arcwise dense.

Generic chains

Definition 3.1

An **order arc** in $\mathrm{Cont}(X)$ is a subcontinuum $\mathcal{A} \subseteq \mathrm{Cont}(X)$ homeomorphic to $[0,1]$ such that for every pair of points $A, B \in \mathcal{A}$, either $A \subseteq B$ or $B \subseteq A$. We denote the space of order arcs by

$$\mathsf{OA}(X) \subseteq \mathrm{Cont}(\mathrm{Cont}(X)).$$

Definition 3.2

A **maximal order arc** in $\mathrm{Cont}(X)$ is an order arc starting from a set $\{x\}$ for some $x \in X$ and ending in X. We can restrict $\mathsf{OA}(X)$ space to the collection of maximal order arcs, denoted by $\mathsf{MOA}(X)$, which is a Polish space.

Definition 3.3

Let X be a continuum, we say that

- GCHEC holds for X if and only if

$$\{\mathcal{C} \in \mathrm{MOA}(X) \mid \forall C \in \mathcal{C}, C \text{ nondegenerate implies } C \simeq X\}$$

  is a comeager subset of $\mathrm{MOA}(X)$.

- GCGHEC holds for X if and only if

$$\{\mathcal{C} \in \mathrm{MOA}(X) \mid \forall^* C \in \mathcal{C}\ (C \simeq X)\}$$

  is a comeager subset of $\mathrm{MOA}(X)$.

Disintegration Theorem

Theorem 3.4

If Z and Y are Polish spaces and $f : Z \to Y$ in a continuous and comeager way, then a set S with Baire property is comeager in Z if and only if $S \cap f^{-1}(y)$ is comeager in $f^{-1}(y)$ for comeager many y in Y.

GCGHEC holds for W_M

Let

$$\mathscr{G} = \left\{ K \in \mathrm{Cont}(W_M) \;\middle|\; \begin{array}{c} K \simeq W_M \text{ and if } b \in \mathcal{B}(W_M) \cap K, \\ \text{then } b \in \mathcal{B}(K) \text{ and it is maximal in } W_M \end{array} \right\}$$

Proposition 3.5

GCGHEC holds for W_M.

Proof idea: Prove that

$$\mathcal{Z}' = \{(\mathcal{C}, K) \in \mathcal{Z} \mid K \in \mathscr{G}\}$$

is comeager in

$$\mathcal{Z} = \{(\mathcal{C}, K) \in \mathrm{MOA}(W_M) \times \mathrm{Cont}(W_M) \mid K \in \mathcal{C}\}$$

and apply the Disintegration Theorem using the projection

$$\pi_2 : \mathcal{Z} \to \mathrm{MOA}(W_M).$$

GCHEC holds for W_M

Lemma 3.6

If $\mathcal{C} \in \mathrm{MOA}(W_M)$ and the collection of $K \in \mathcal{C}$ with $K \simeq W_M$ is dense in $\mathcal{C}$, then every nondegenerate element of $\mathcal{C}$ is homeomorphic to W_M.

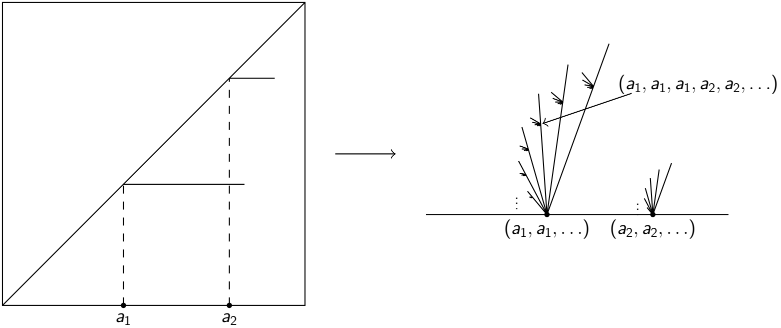
Theorem 3.7

GCHEC holds for W_M.

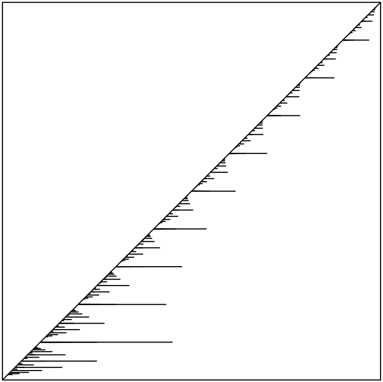
Properties of the comeager maximal order arc in $\mathrm{MOA}(W_M)$

The description of W_∞ by generalized inverse limits (GIL)

Banič, Črepnjak, Sovič together with more two authors described a way to define the Ważewski universal dendrite as the generalized inverse limit of a single set-valued bonding function $f : [0,1] \to \mathrm{Fin}[0,1]$.



A chain using the GIL



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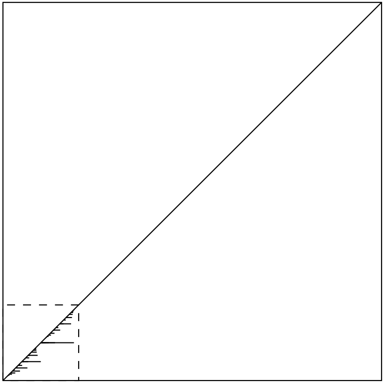

$$f_t^{-1}(s) = \begin{cases} [a_n, a_n + (t - a_n)(b_n - a_n)/(1 - a_n)], & \text{if } s = a_n \text{ for some } n \in \mathbb{N}, \\ \{s\}, & \text{otherwise.} \end{cases}$$

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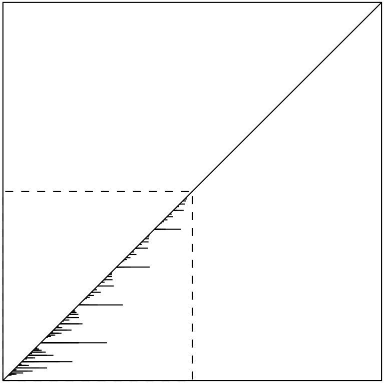

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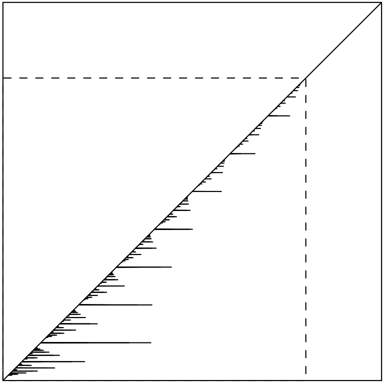

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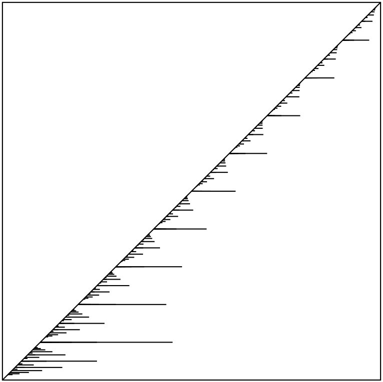

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Theorem 4.1

The generic element $\mathcal{C}$ of $\mathrm{MOA}(W_M)$ satisfies:

(i) If $K \in \mathcal{C}$ is nondegenerate, then $K \simeq W_M$;

(ii) $\cap\mathcal{C} = \{x\}$ where $x \in End(W_M)$;

(iii) If $K_1, K_2 \in \mathcal{C}$ with $K_1 \subsetneq K_2$, then K_1 is nowhere dense in K_2.

(iv) If $K \in \mathcal{C}$, then $\mathrm{End}(K) \setminus (\cup\{K' \in \mathcal{C} \mid K' \subsetneq K\})$ is dense in K.

(v) If $K_1, K_2 \in \mathcal{C}$ with $K_1 \subsetneq K_2$, then every branching point of K_1 is maximal in K_2.

(vi) If $K_1, K_2 \in \mathcal{C}$ with $K_1 \subsetneq K_2$, then $|r^{-1}_{K_2,K_1}(x)| > 1$ implies $x \in \mathrm{End}(K_1)$.

Muito obrigado!