External q-hyperconvexity in T_0-quasi-metric spaces

Collins Amburo Agyingi

Department of Mathematical Sciences

University of South Africa (UNISA)

38TH SUMMER CONFERENCE ON TOPOLOGY AND ITS APPLICATIONS
(COIMBRA, PORTUGAL)

09 July 2024

Outline

Quasi-pseudometric

- Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the non-negative reals.Then d is called a **quasi-pseudometric** on X if
  (a) $d(x, x) = 0$ whenever $x \in X$, and
  (b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, x \in X$.
  The pair (X, d) is said to be a quasi-pseudometric space.
- If d satisfies the additional condition that $d(x, y) = 0 = d(y, x)$ implies that $x = y$, we call d a T_0-quasi-metric. The set X together with a T_0-quasi-metric is called a T_0-quasi-metric space.

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Conjugate quasi-pseudometric

- If d is a quasi-pseudometric on a set X, then we define the conjugate quasi-pseudometric $d^{-1} : X \times X \to [0, \infty)$ by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$.
- If d is a T_0-quasi-quasi-metric, then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a metric.
- Let (X, d) be a quasi-pseudometric space. By an open ϵ-ball centered at a point $x \in X$ denoted $B_d(x, \epsilon)$, we mean $\{y \in X : d(x, y) < \epsilon\}$ for every $\epsilon > 0$.
- On the other hand $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is known as the closed ϵ-ball centered at $x \in X$ for some $\epsilon \geq 0$.

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Example and remark

Example

Given two real numbers a and b we shall write $a \dot{-} b$ *for* $\max\{a - b, 0\}$ *which we can also denote by* $(a - b) \vee 0$. *Note that* $d(x, y) = x \dot{-} y$ *with* $x, y \in \mathbb{R}$ *defines a* T_0*-quasi-metric on the set* $\mathbb{R}$ *of the reals. Observe that* $x \mapsto -x$ *defines a bijective isometric map from* $(\mathbb{R}, d)$ *to* $(\mathbb{R}, d^t)$.

Remark

The collection $\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$ *of all "open" balls yields a base for a topology* $\tau(d)$. *It is called the topology induced by d on X.*

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Family of double balls

For a quasi-pseudometric space (X, d), the pair $(C_d(x, r); C_{d^{-1}}(x, s))$ with $x \in X$ and non-negative reals r and s will be called a double ball at x. We talk of a family $[(C_d(x_i, r_i))_{i \in I}; (C_{d^{-1}}(x_i, s_i))_{i \in I}]$ of double balls, with $x_i \in X$ and $r_i, s_i \geq 0$ whenever $i \in I$.

Let us denote by $\mathcal{P}_0(X)$ the set of all nonempty subsets of X. Given $A \in \mathcal{P}_0(X)$, we define

$$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\},$$

whenever $x \in X$.

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Definition of q-hyperconvexity

Definition

A(n extended) quasi-pseudometric space (X, d) will be called q-hyperconvex provided that for each family $(x_i)_{i\in I}$ of points in X and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of non-negative real numbers the following condition holds:

$$\text{If } d(x_i, x_j) \leq r_i + s_j \text{ whenever } i, j \in I,$$

then

$$\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

Given a quasi-pseudometric space (X, d) with $x \in X$ and non-negative real numbers r and s, we shall make use of the following notation

$$C_x(r, s) := C_d(x, r) \cap C_{d^{-1}}(x, s).$$

Definition

A subset D of a quasi-pseudometric space (X, d) is called bounded if there is a real number $M > 0$ such that $d(x, y) < M$ for every $x, y \in D$.

See immediately then that a subset D of (X, d) will be said to be bounded if and only if there are $x \in X$ and non-negative real numbers r and s such that $D \subseteq C_x(r, s)$.

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Q-admissible subset of a quasi-pseudometric space (X, d)

Definition

Let (X, d) be a quasi-pseudometric space. A nonempty bounded subset of X that can be written as the intersection of a nonempty family of sets of the form $C_x(r, s)$ where r and s are non-negative real numbers and $x \in X$, will be called q-admissible.

We shall denote by $\mathcal{A}_q(X)$ the set of q-admissible subsets of X.

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Definition of external q-hyperconvex subset

Definition

Let (X, d) be a quasi-pseudometric space. A subspace E of (X, d) is said to be externally q-hyperconvex (relative to X) if given any family $(x_i)_{i \in I}$ of points in X and families $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ of non-negative real numbers the following condition holds:

$$\text{If } d(x_i, x_j) \leq r_i + s_j \text{ whenever } i, j \in I,\ dist(x_i, E) \leq r_i, dist(E, x_i) \leq s_i$$

whenever $i \in I$, then

$$\bigcap_{i \in I} C_{x_i}(r_i, s_i) \cap E \neq \emptyset.$$

We shall denote by $\mathcal{E}_q(X, d)$ the set of nonempty externally q-hyperconvex subsets of a quasi-pseudometric space (X, d).

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Proposition

Let (X,d) be a quasi-pseudometric space and $E \subseteq X$.

(a) *If $E \in \mathcal{E}_q(X,d)$, then $E \in \mathcal{E}_q(X,d^{-1})$.*

(b) *If $E \in \mathcal{E}_q(X,d)$, then E is an externally hyperconvex subspace of (X,d^s).*

It is easy to prove that if X is q-hyperconvex, then $\mathcal{A}_q(X) \subset \mathcal{E}_q(X,d)$.

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Example

Let $X = \mathbb{R}$ be the set of reals equipped with the T_0-quasi-metric u defined by $u(x, y) := x \dot{-} y = \max\{x - y, 0\}$. Then (X, u) is q-hyperconvex by [2, Example 1]. The subset $A = [-1, 2]$ is externally q-hyperconvex (relative to $\mathbb{R}$) since $A \in \mathcal{A}_q(X)$, that is, $A = C_u(0, 1) \cap C_{u^{-1}}(0, 2)$.

Our first result is the following.

Proposition

Any pairwise intersecting finite collection of externally q-hyperconvex subsets of a q-hyperconvex T_0-quasi-metric space has a nonempty intersection and this intersection is also externally q-hyperconvex.

Remark

What about a countable collection? By imposing additional conditions, we obtain a similar result.

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Lemma

([3, Theorem 6.5]) Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space. Moreover, let $(X_i)_{i\in I}$ be a descending family of non-empty externally q-hyperconvex subsets of X, where we assume that I is a chain such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ hold if and only if $X_{i_2} \subseteq X_{i_1}$. Then,

$$\emptyset \neq \bigcap_{i\in I} X_i \in \mathcal{E}_q(X, d).$$

Proposition

Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space and $\{A_i\}_{i\in\mathbb{N}}$ be a countable family of pairwise intersecting externally q-hyperconvex subsets of X. Then

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Lemma

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An arbitrary collection of externally q-hyperconvex subsets?

Proposition

Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space and $\{A_i\}_{i\in I}$ be any family of pairwise intersecting externally q-hyperconvex subsets such that at least one of them is bounded. Then

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We end with the following proposition.

Proposition

Let (X,d) be a T_0-quasi-metric space and $Y \subseteq X$ be such that $Y \in \mathcal{E}_q(X,d)$. Moreover, let A be externally q-hyperconvex (relative to Y). Then $A \in \mathcal{E}_q(X,d)$.

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