The Specification Property of Homeomorphisms on Tree-like Continua

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Motivation and Definitions

A *continuum* is a compact connected metric space.

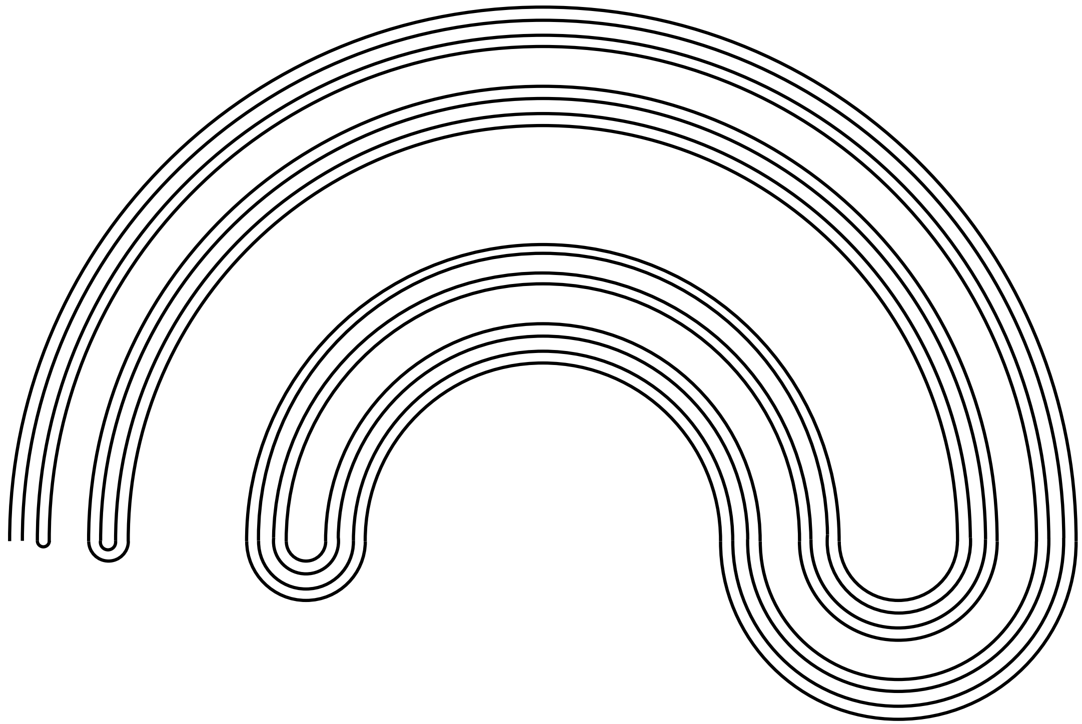
The primary motivation to study the necessary topology of continua that admit homeomorphisms with particular "chaotic" properties.
In particular, I am interested in what conditions for the continuum to be

1. indecomposable
2. contain an indecomposable subcontinuum
3. if hereditarily decomposable, then what can be said about the topology

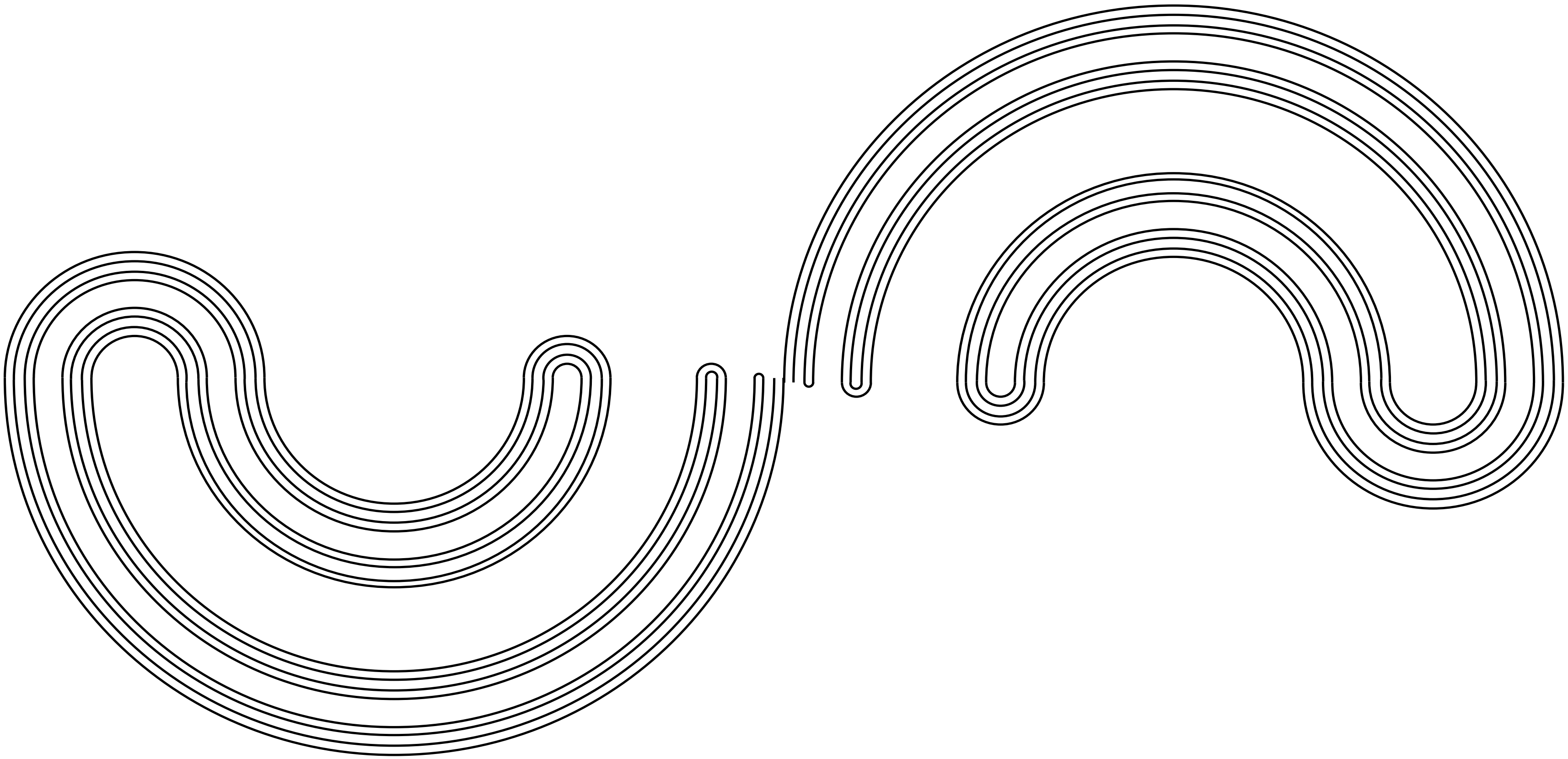
A continuum is *indecomposable* if every proper subcontinuum has empty interior.

A continuum X is $1/n$ *indecomposable* if whenever $\{A_i\}_{i=1}^n$ are a collection of pairwise disjoint subcontinua, at least one of $\{A_i\}_{i=1}^n$ has empty interior.

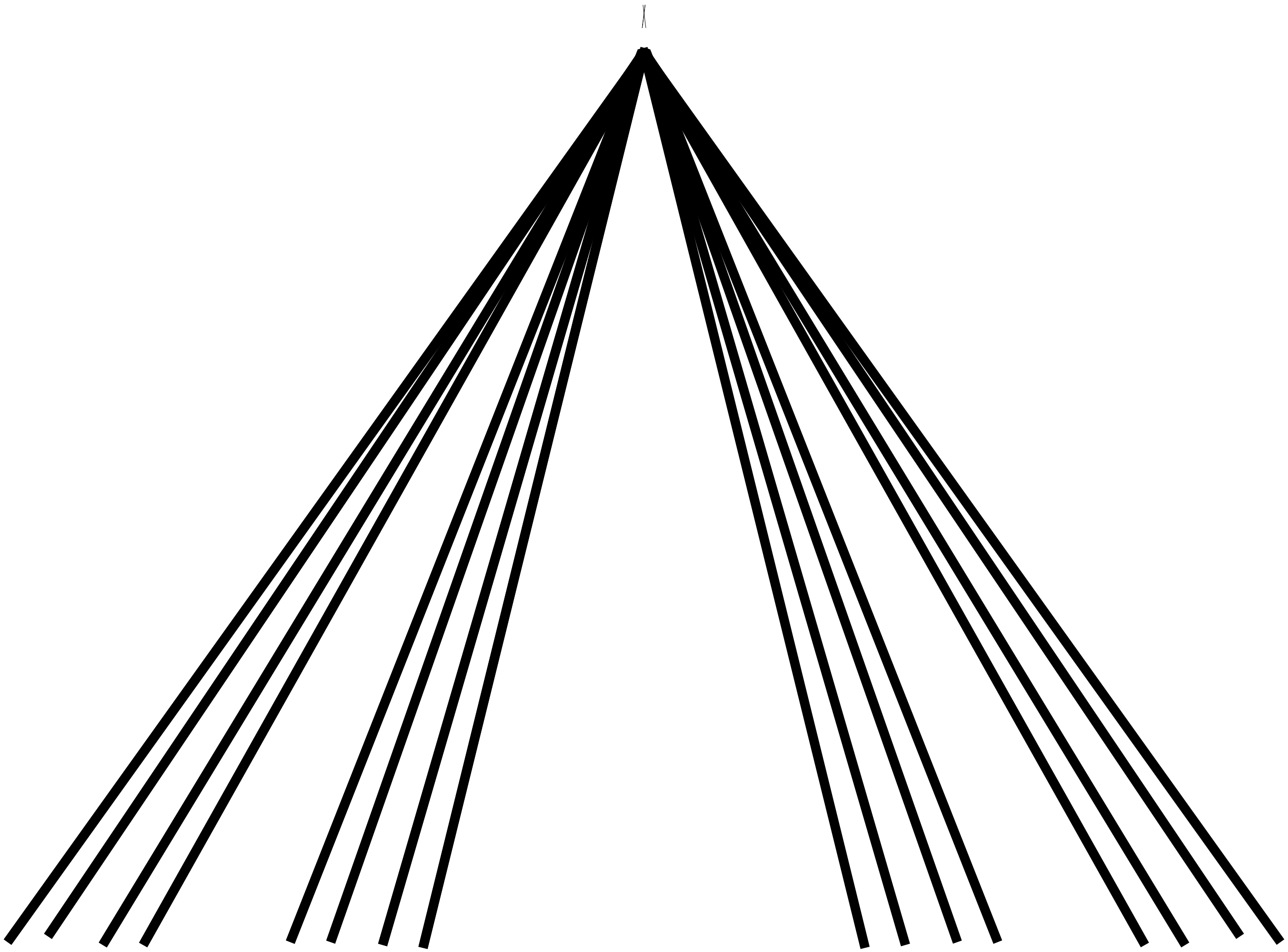
Note: $1/2$ indecomposable is also called *semi-indecomposable*.



The buckethandle continuum is indecomposable.



A $1/2$ or semi-indecomposable continuum. It is the union of two indecomposable subcontinua



The Cantor Fan is $1/2$ or *semi*- indecomposable. However, it does not contain an indecomposable subcontinuum. Hence, it is hereditarily decomposable.

Theorem

If $f : X \longrightarrow X$ is a map of a continuum then:

$$
\begin{array}{rc}
 & f \text{ is fully continuum-wise expansive.} \\
 & \Downarrow \\
f \text{ has the specification property} \Longrightarrow & f \text{ is mixing.} \\
 & \Downarrow \\
 & f \text{ is weakly mixing.} \\
 & \Downarrow \\
 & f \text{ is totally transitive.} \\
 & \Downarrow \\
 & f \text{ is transitive.} \\
 & \Downarrow \\
 & f \text{ has sensitive dependence on initial conditions.}
\end{array}
$$

A map f is *positively continuum-wise fully expansive* if for every $\epsilon > 0$ and nondegenerate subcontinuum Y, there exists $N = N(\epsilon, Y)$ such that $\mathrm{d}_H(f^n(Y), X) < \epsilon$ for all $n \geq N$.

A map f is *mixing* if for every open sets U, V of X, there exists an M such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq M$.

That is, if U is open, then $\mathrm{d}_H(f^n(U), X) \to 0$ as $n \to \infty$.

Let $f : X \longrightarrow X$ be a function on metric space X. (f, X) has the *Specification Property* if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that for any finite set of points $x_1, ..., x_m \in X$ and a a set of position integers $n_1, n_2, ..., n_m$ there exists an $y \in X$ such that

$$d(f^j(y), f^j(x_1)) < \epsilon \text{ for every } j \in \{0, ..., n_1]$$

$$d(f^j(y), f^j(x_2)) < \epsilon \text{ for every } j \in \{n_1 + N, ..., n_1 + n_2 + N\}$$

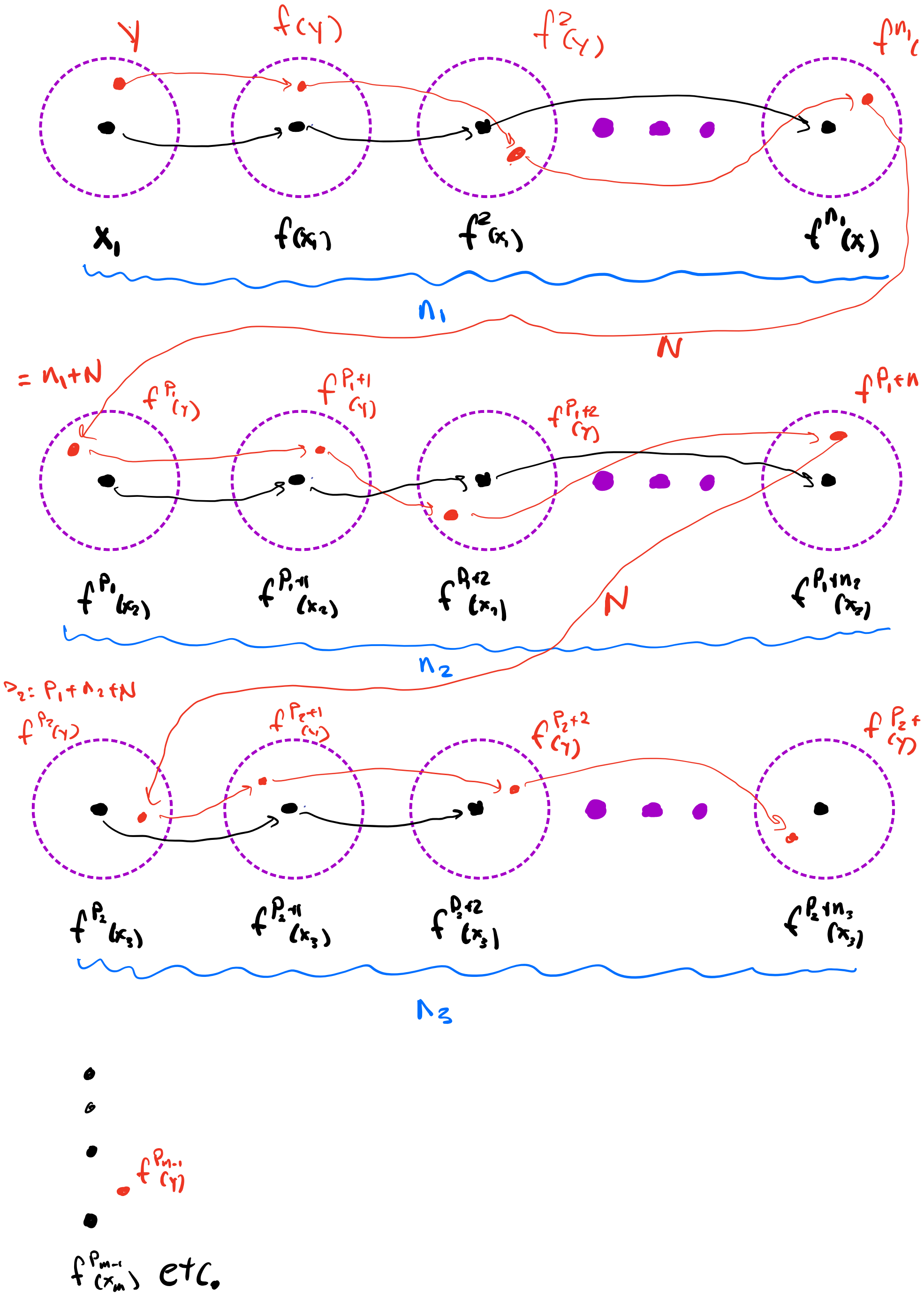
$$.$$

$$.$$

$$.$$

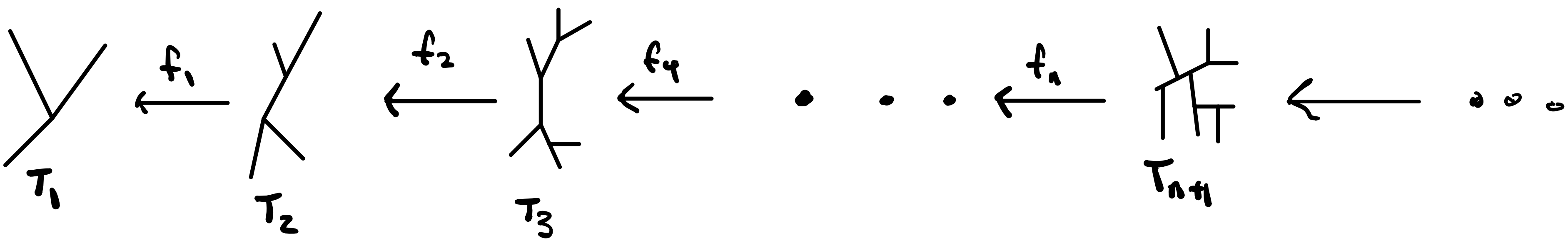
$$d(f^j(y), f^j(x_n)) < \epsilon \text{ for every}$$

$$j \in \{n_1 + ... + n_{m-1} + (m-1)N, n_1 + ... + n_{m-1} + n_m + (m-1)N\}.$$


y
f(y)
$f^2(y)$
x_1
$f(x_1)$
$f^2(x_1)$
$f^{n_1}(x_1)$
n_1
N
$= n_1+N$
$f^{p_1}(y)$
$f^{p_1+1}(y)$
$f^{p_1+2}(y)$
$f^{p_1}(x_2)$
$f^{p_1+1}(x_2)$
$f^{p_1+2}(x_2)$
$f^{p_1+n_2}(x_2)$
n_2
N
$p_2 = p_1+n_2+N$
$f^{p_2}(y)$
$f^{p_2+1}(y)$
$f^{p_2+2}(y)$
$f^{p_2}(x_3)$
$f^{p_2+1}(x_3)$
$f^{p_2+2}(x_3)$
$f^{p_2+n_3}(x_3)$
n_3
$f^{p_{m-1}}(y)$
$f^{p_{m-1}}(x_m)$ etc.

A continuum is *tree-like* if it is the inverse limit of trees.
That is let $f_i : T_{i+1} \longrightarrow T_i$ be a collection of maps of tree graphs $\{T_i\}_{i=1}^{\infty}$. Then the inverse limit of $\{T_i, f_i\}$ is a continuum $\widehat{X}$

$$\widehat{X} = \varprojlim \{T_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} T_i | f_i(x_{i+1}) = x_i\}.$$



If $\mathbf{x} = \langle x_i \rangle_{i=1}^{\infty}$ and $\mathbf{y} = \langle y_i \rangle_{i=1}^{\infty}$ are two points of the inverse limit, we define distance to be

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_{T_i}(x_i, y_i)}{2^i}.$$

Theorem

(Kato) If X is a continuum that admits a positively fully continuum-wise expansive homeomorphism then X is indecomposable.

The following theorems are by Jorge Martínez-Montejano and Verónica Martínez-de-la-Vega and myself.

Theorem

(M,M,MV) If X is a G-like continuum that admits a mixing homeomorphism, then X is indecomposable.

A continuum is G-like if it is the inverse limit of the same graph G.

Theorem

(M,M,MV) If X is a tree-like continuum that admits a mixing homeomorphism, then X is semi-indecomposable.

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Theorem

(M,M,MV) If X is a tree-like continuum that admits a mixing homeomorphism, then X is semi-indecomposable.

Also, we have:

Theorem

(M,M,MV) There exists a hereditarily decomposable tree-like continuum that admits a mixing homemorphism.

Recently by Iztok Banič, Goran Erceg, Judy Kennedy, Van Nall and myself:

Theorem

(B,E,K,M,N) There exists an uncountable collection of non-homeomorphic smooth fans that admit mixing homeomorphisms.

Included in these are the Cantor Fan and the Lelek Fan. Also, we have

Theorem

(B,E,K,M,N) There exists an uncountable collection of non-homeomorphic non-smooth fans that admit mixing homeomorphisms.

Recap:

Theorem

If $f : X \longrightarrow X$ is a map of a tree-like continuum then:

$$f \text{ is fully continuum-wise expansive} \implies X \text{ is indecomposable}$$

f *has the specification property*

$\Downarrow$

f *is mixing* $\implies$ X *is semi-indecomposable.* X *can be a fan. If* X *is* G*-like then* X *is indecomposable.*

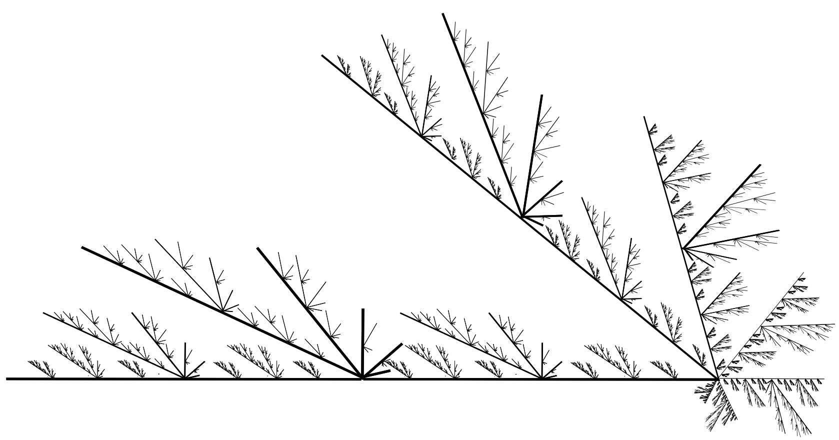
Main result

Theorem

There exists a hereditarily decomposable tree-like continuum that admits a homeomorphism with the specification property.

Note: This shows that the specification property does not imply fully continuum-wise expansive.

My construction is by using the shift homeomorphism on the inverse limit of universal dendrites with the same bonding map that itself has the specification property.



The universal dendrite D_ω.

Main result

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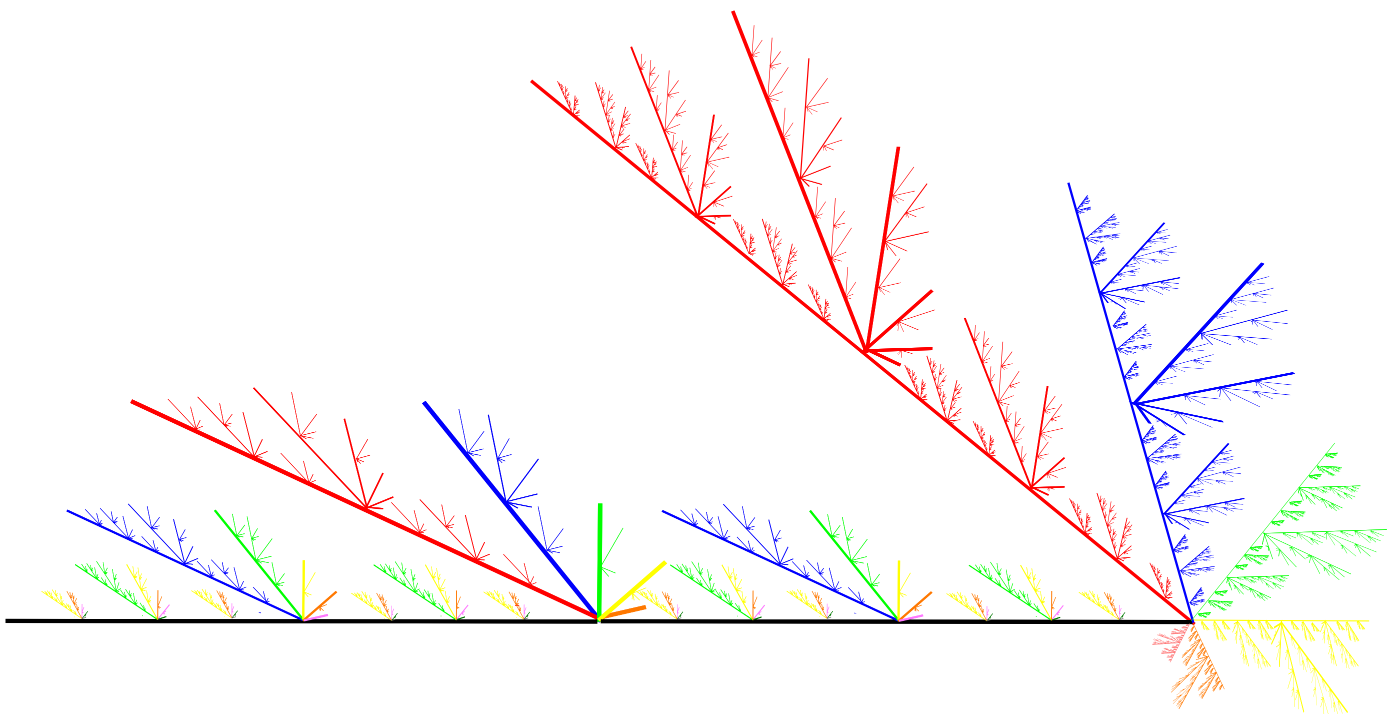
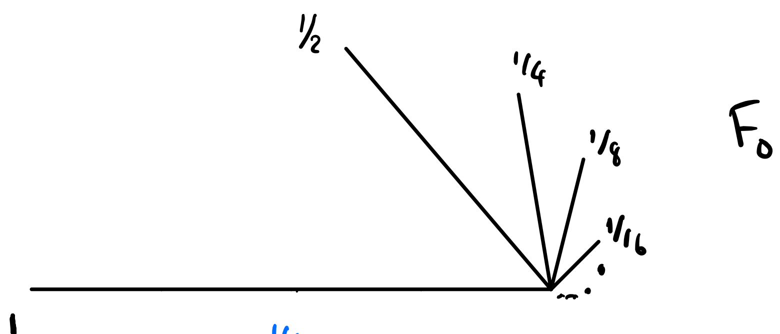
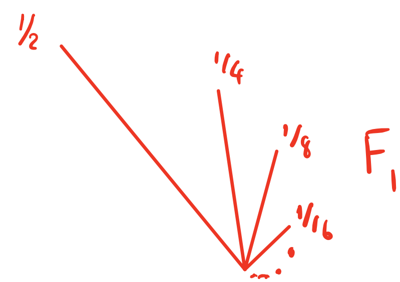


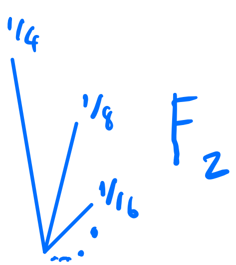
The universal dendrite D_ω.

Let $[0, \frac{1}{2^n}]_n$ be an arc of length $\frac{1}{2^n}$ for $n \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$.
Let $F_k = \bigcup_{n=k}^{\infty} [0, \frac{1}{2^n}]_n / \{0\}$ be a "harmonic" fan of size k (Diameter is $\frac{1}{2^k}$).
The identification point 0 in F_k is called the *root of* F_k. Let $\overline{0} = (0, 0, 0, ...)$. We denote a point $\mathbf{x} \in F_k$ by

$$\mathbf{x} = \begin{cases} (k, \overline{0}) & \text{if } \mathbf{x} \text{ is the root of } F_k \\ (n_0, r_0, \overline{0}) & \text{if } r_0 \in (0, \frac{1}{2^{n_0}}]_{n_0} \end{cases}$$







Now let $\mathcal{D}(a,b)$ be the dyadic rationals on (a,b), that is

$$\mathcal{D}(a,b) = \{\frac{p}{2^n} \in (a,b) \mid P \text{ is an odd integer and } n \in \mathbb{N}_0\}.$$

Let $D_0 = F_0$, with $\overline{0}$ as the root and for each $(n_0, \frac{p_0}{2^{k_0}}, \overline{0})$, where p_0 is odd, in $[0, \frac{1}{2^{n_0}}]_n$, identify the root from a copy of F_{k_0} to $(n_0, \frac{p_0}{2^{k_0}}, \overline{0})$.

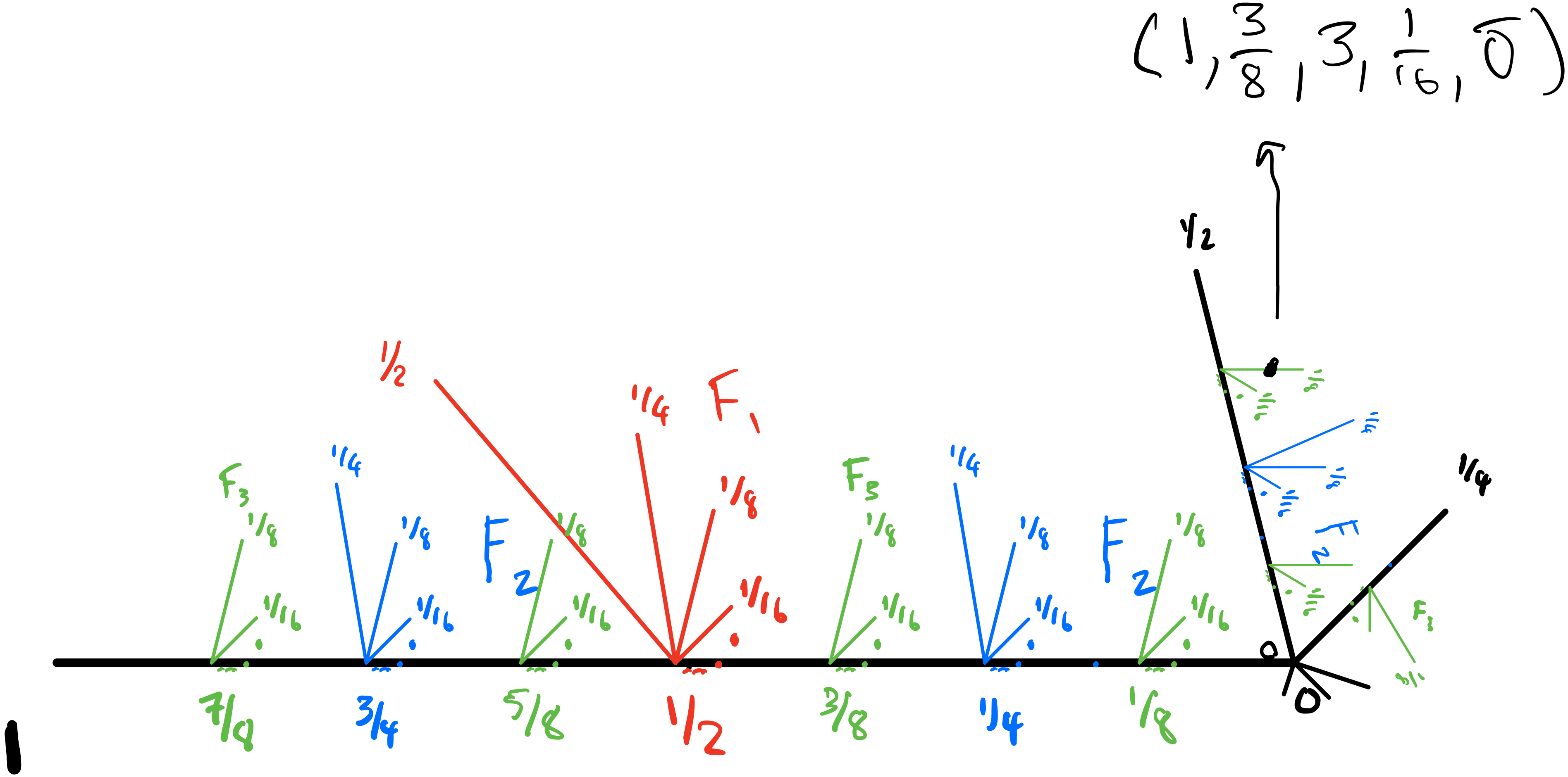
Define the *dyadic points for* D_0 by

$$\mathcal{D}(D_0) = \{(n_0, r_0, \overline{0}) \in D_0 \mid r_0 \in \mathcal{D}(0,1)\}$$

and let

$$D_1 = F_0 \cup \bigcup_{n \in \mathbb{N}_0} \bigcup_{\frac{p_0}{2^{k_0}} \in \mathcal{D}(D_0)} F_k$$

with the above identification.


(1, 3/8, 3, 1/16, 0̄)
1/2
1/4
F₁
F₂
F₃
1/8
1/16
1
7/8
3/4
5/8
1/2
3/8
1/4
1/8
0

We denote a point $\mathbf{x} = (n_0, \frac{p_0}{2^{n_1}}, n_1, r_1, \overline{0}) \in D_1 - D_0$ where $(n_1, r_1, \overline{0})$ is in a copy of F_{n_1} whose root is identified to $(n_0, \frac{p_0}{2^{n_1}}, \overline{0})$ in D_0.

Continuing inductively, suppose that $D_0, D_1, ..., D_m$ have been found. Then for each $(n_0, \frac{p_0}{2^{n_1}}, n_1, \frac{p_1}{2^{n_2}}, , ..., n_m, \frac{p_m}{2^{n_{m+1}}}, \overline{0}) \in D_m - D_{m-1}$, identify the root from a copy of $F_{n_{m+1}}$.

Define *the dyadic points for* $D_m - D_{m-1}$ by $\mathcal{D}(D_m) = \{(n_0, \frac{p_0}{2^{n_1}}, ..., n_m, \frac{p_m}{2^{n_{m+1}}}, \overline{0}) \in D_m - D_{m-1} \mid r_m \in \mathcal{D}(0,1)\}$. Let

$$D_{m+1} = D_m \cup \bigcup_{n \geq m} \bigcup_{\mathbf{x} \in \mathcal{D}(D_m)} F_{n_{m+1}}$$

We denote a point

$$\mathbf{x} = (n_0, \frac{p_0}{2^{n_1}}, \dots, n_m, \frac{p_m}{2^{n_{m+1}}}, n_{m+1}, r_{m+1}, \overline{0}) \in D_{m+1} - D_m$$

where $(n_{m+1}, r_{m+1}, \overline{0})$ is in a copy of $F_{n_{m+1}}$ whose root is identified to $(n_0, \frac{p_0}{2^{n_1}}, \dots, n_m, \frac{p_m}{2^{n_{m+1}}}, \overline{0})$ in $D_m - D_{m-1}$.

Note: if r_m is not a dyadic rational, then r_m must be followed by $\overline{0}$.

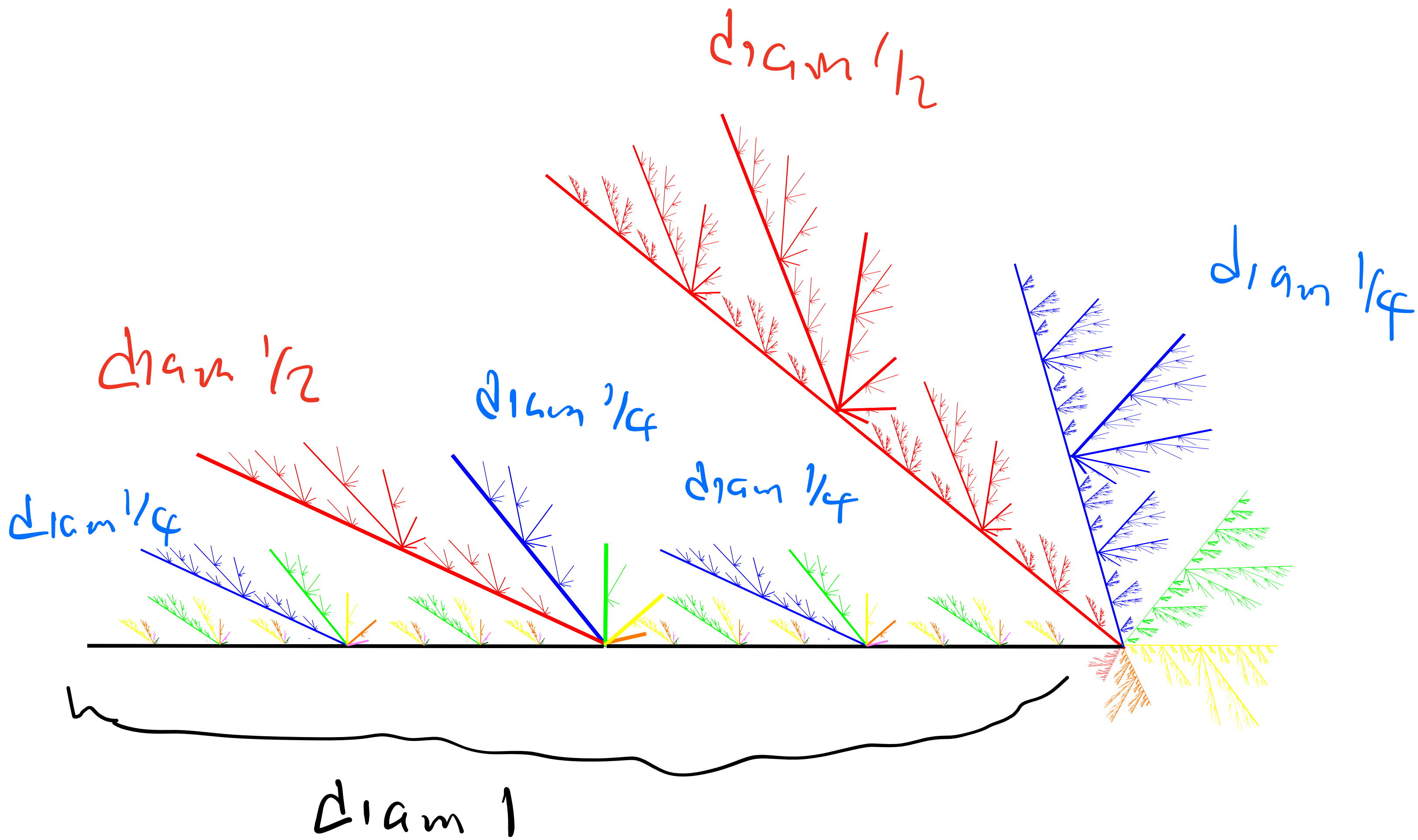
Let $D_\omega = \overline{\bigcup_{i=0}^{\infty} D_i}$. If $\mathbf{x} \in D_\omega - \bigcup_{i=0}^{\infty} D_i$, then $\mathbf{x}$ is any sequence of the form

$$(n_0, \frac{p_0}{2^{n_1}}, n_1, \frac{p_1}{2^{n_2}}, ..., n_m, \frac{p_m}{2^{n_{m+1}}} ...)$$

where p_i is odd and $n_0 < n_1 < ...n_m < ....$

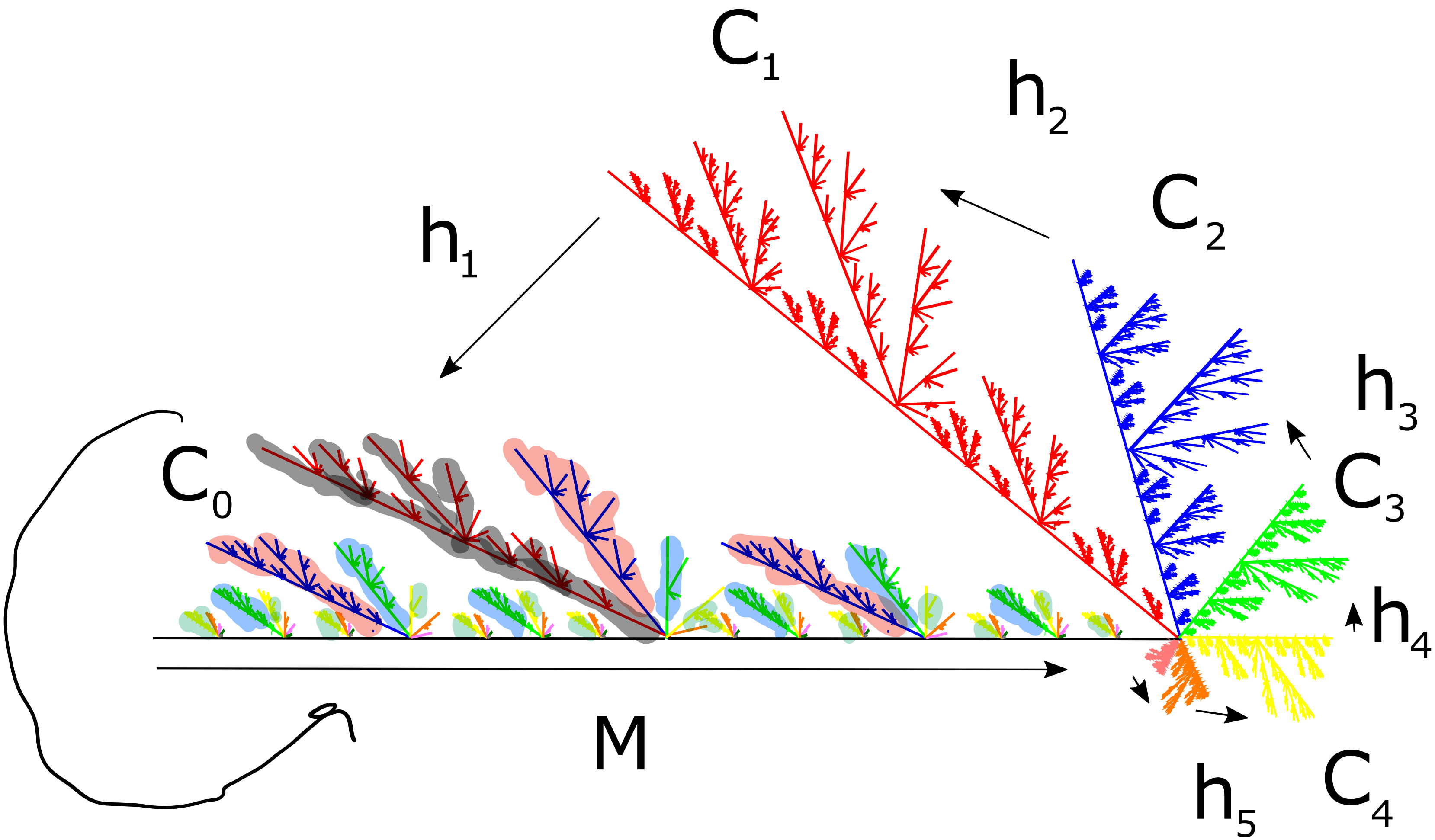
If $\mathbf{x} = (n_0, r_0, ..., n_k, r_k, \overline{0})$ and $\mathbf{y} = (m_0, s_0, ..., m_k, s_k, \overline{0})$. Let p be the first index that $\mathbf{x}$ and $\mathbf{y}$ differ. Then we define the distance in D_ω by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_{i=p}^{\infty} r_i + \sum_{i=p}^{\infty} s_i & \text{if } n_p \neq m_p \\ |r_p - s_p| + \sum_{i=p}^{\infty} r_i + \sum_{i=p}^{\infty} s_i & \text{if } r_p \neq s_p \end{cases}$$



The universal dendrite D_ω.

Map of D_ω with the specification property



$\sigma : D_\omega \longrightarrow D_\omega$ has the specification property.

Now, define $\sigma : D_\omega \longrightarrow D_\omega$ by

$$\sigma(\overline{0}) = \overline{0},$$

for $\mathbf{x} \in D_k$:

$$\sigma(n_0, r_0, ..., n_k, r_k, \overline{0}) = \begin{cases} (n_0 - 1, 2r_0, ..., n_k - 1, 2r_k, \overline{0}) & \text{if } n_0 > 0 \\ (n_1 - 1, 2r_1, ..., n_{k+1} - 1, 2r_{k+1}, \overline{0}) & \text{if } n_0 = 0 \end{cases}$$

and for $\mathbf{x} \in D_\omega - \bigcup_{i=0}^{\infty} D_i$:

$$\sigma(n_0, r_0, ..., n_k, r_k, ...) = \begin{cases} (n_0 - 1, 2r_0, ..., n_k - 1, 2r_k, ...) & \text{if } n_0 > 0 \\ (n_1 - 1, 2r_1, ..., n_{k+1} - 1, 2r_{k+1}, ...) & \text{if } n_0 = 0 \end{cases}$$

For $\mathbf{x} = (n_0, r_0, ..., n_k, r_k, \overline{0})$ define

$$p(\mathbf{x}, m) = \min\{i \mid n_i \geq m\} \text{ and } q(\mathbf{x}, m) = \max\{i \mid n_i < m\}.$$

Let $p = p(\mathbf{x}, m)$ then it follows that

$\sigma^m(n_0, r_0, ..., n_k, r_k, \overline{0}) =$

$$\begin{cases} (n_p - m, 2^m r_0, ..., n_k - m, 2^m r_k, \overline{0}) & \text{if } n_k \geq m \\ (\overline{0}) & \text{if } n_k < m \end{cases}$$

Also

$$\sigma^m(n_0, r_0, ..., n_k, r_k, ...) = (n_p - m, 2^m r_0, ..., n_k - m, 2^m r_k, ...).$$

Let $\epsilon > 0$ and $N \in \mathbb{N}$ be such that $\frac{1}{2^N} \leq \frac{\epsilon}{4} < \frac{1}{2^{N-1}}$.
Let

$$\mathbf{x}_1 = (n_0^1, r_0^1, ..., n_k^1, r_k^1, ...)$$

$$\mathbf{x}_2 = (n_0^2, r_0^2, ..., n_k^2, r_k^2, ...)$$

$$\vdots$$

$$\mathbf{x}_m = (n_0^m, r_0^m, ..., n_k^m, r_k^m, ...)$$

and $n_1, n_2, ..., n_m$ be positive integers.

Let

$$p_1 = 0 \qquad , \quad q_1 = q(\mathbf{x}_1, n_1 + N)$$

$$p_2 = p(\mathbf{x}_2, n_1 + N) \ , \quad q_2 = q(\mathbf{x}_2, n_1 + n_2 + N)$$

$$.$$

$$.$$

$$.$$

$$p_m = p(\mathbf{x}_m, n_1 + ... n_{m-1} + (m-1)N), \ q_m = q(\mathbf{x}_m, n_1 + ... + n_m + (m)N)$$

Now define

$$\mathbf{y} = (n^1_{p_1}, r^1_{p_1}, ..., n^1_{q_1}, \widehat{r}^1_{q_1}, n^2_{p_2}, r^2_{p_2}, ..., n^2_{q_2}, \widehat{r}^2_{q_2}, ..., n^m_{p_m}, r^m_{p_m}, ..., n^m_{q_m}, \widehat{r}^m_{q_m}, \widehat{0})$$

where $\widehat{r}^i_{q_i} = r^i_{q_i}$ if $r^i_{q_i}$ is a dyadic rational and $\widehat{r}_{q_i}$ is a the dyadic rational of the form $\frac{i}{2^{n^i_{q_i}+1}}$ closest to $r^i_{q_i}$ if it is not a dyadic rational. If p_i does not exist move to p_{i+1}. If p_m does not exist, then finish with $\overline{0}$.
This $\mathbf{y}$ will have the property that

$$d(f^j(\mathbf{y}), f^j(\mathbf{x}_1)) < \epsilon \text{ for every } j \in \{0, ..., n_1]$$

$$d(f^j(\mathbf{y}), f^j(\mathbf{x}_2)) < \epsilon \text{ for every } j \in \{n_1 + N, ..., n_1 + n_2 + N\}$$

.

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$$d(f^j(\mathbf{y}), f^j(\mathbf{x}_n)) < \epsilon \text{ for every}$$

$$j \in \{n_1 + ... + n_{m-1} + (m-1)N, n_1 + ... + n_{m-1} + n_m + (m-1)N\}.$$

Let $\epsilon = 1/4$ So $N = 4$ since $1/2^4 = \epsilon/4$. Let

$$\mathbf{x}_1 = (3, 3/2^4, 4, 21/2^7, 7, 11/100, \bar{0})$$

$$\mathbf{x}_2 = (2, 1/2^3, 3, 3/2^5, 5, \pi/1000, \bar{0})$$

$$\mathbf{x}_3 = (11, 3/2^{12}, 15, 21/2^{19}, 19, 11/2^{26}, 26, 1/2^{29}, , , \ldots)$$

and $n_1 = 3$, $n_2 = 4$, and $n_3 = 3$.

$n_1 + N = 7$, $p_1 = 0$, $q_1 = 1$

$n_1 + n_2 + 2N = 15$ p_2, q_2 DNE

$n_1 + n_2 + n_3 + 3N = 22$ $p_3 = 1$, $q_3 = 2$

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and $n_1 = 3$, $n_2 = 4$, and $n_3 = 3$. Let

$$\mathbf{y} = (3, 1/2^4, 4, 11/2^7, 15, 13/2^{19}, 19, 11/2^{26}, \bar{0})$$

$$\sigma(\mathbf{x}_1) = (2, 1/2^3, 3, 11/2^6, 6, 11/50, \bar{0})$$

$$\sigma(\mathbf{y}) = (2, 1/2^3, 3, 11/2^6, 6, 11/50, 14, 13/2^{18}, 18, 11/2^{25}, \bar{0})$$

$$\sigma^3(\mathbf{x}_1) = (0, 1/2^1, 1, 11/2^4, 4, 44/50, \bar{0})$$

$$\sigma^3(\mathbf{y}) = (0, 1/2^1, 1, 11/2^4, 4, 44/50, 12, 13/2^{16}, 16, 11/2^{23}, \bar{0})$$

$n_1 + N = 7$, $p_1 = 0$, $q_1 = 1$

$n_1 + n_2 + 2N = 15$ p_2, q_2 DONE

$n_1 + n_2 + n_3 + 3N = 22$ $p_3 = 1$, $q_3 = 2$

$$\mathbf{x}_1 = (3, 1/2^4, 4, 11/2^7, 7, 11/100, \bar{0})$$

$$\mathbf{x}_2 = (2, 1/2^3, 3, 3/2^5, 5, \pi/1000, \bar{0})$$

$$\mathbf{x}_3 = (11, 3/2^{12}, 15, 13/2^{19}, 19, 11/2^{26}, 26, 1/2^{29}, , , \dots)$$

and $n_1 = 3$, $n_2 = 4$, and $n_3 = 3$. Let

$$\mathbf{y} = (3, 1/2^4, 4, 11/2^7, 15, 13/2^{19}, 19, 11/2^{26}, \bar{0})$$

$$\sigma^7(\mathbf{x}_2) = (\bar{0})$$

$$\sigma^7(\mathbf{y}) = (8, 13/2^{12}, 12, 11/2^{19}, \bar{0})$$

$$\sigma^{11}(\mathbf{x}_2) = (\bar{0}) \qquad [$$

$$\sigma^{11}(\mathbf{y}) = (4, 21/2^8, 8, 11/2^{15}, \bar{0})$$

$n_1 + N = 7$, $p_1 = 0$, $q_1 = 1$

$n_1 + n_2 + 2N = 15$ p_2, q_2 DONE

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$$\mathbf{x}_1 = (3, 1/2^4, 4, 11/2^7, 7, 11/100, \bar{0})$$

$$\mathbf{x}_2 = (2, 1/2^3, 3, 3/2^5, 5, \pi/1000, \bar{0})$$

$$\mathbf{x}_3 = (11, 3/2^{12}, 15, 13/2^{19}, 19, 11/2^{26}, 26, 1/2^{29}, , , \dots)$$

and $n_1 = 3$, $n_2 = 4$, and $n_3 = 3$. Let

$$\mathbf{y} = (3, 1/2^4, 4, 11/2^7, 15, 13/2^{19}, 19, 11/2^{26}, \bar{0})$$

$$\sigma^{15}(\mathbf{x}_3) = (0, 13/2^4, 4, 11/2^{11}, 11, 1/2^{14}, \dots)$$

$$\sigma^{15}(\mathbf{y}) = (0, 13/2^4, 4, 11/2^{11}, \bar{0})$$

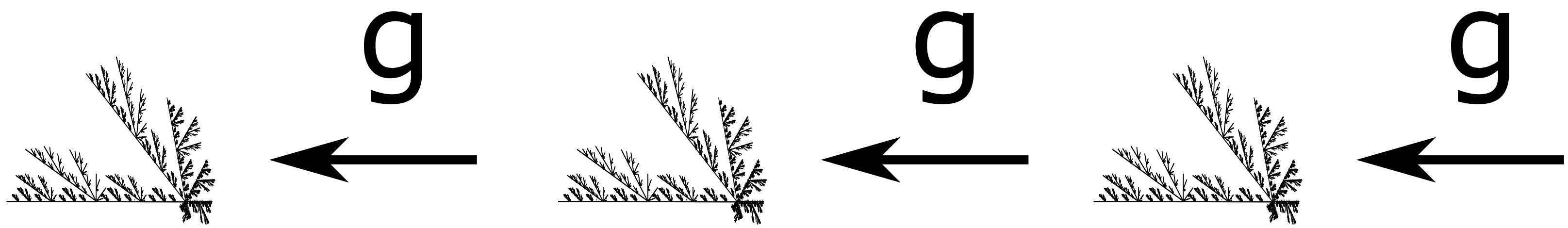
$$\sigma^{18}(\mathbf{x}_3) = (1, 11/2^8, 8, 1/2^{11}, \dots)$$

$$\sigma^{18}(\mathbf{y}) = (1, 11/2^8, 8, 1/2^{11}, \bar{0})$$

$n_1 + N = 7$, $p_1 = 0, q_1 = 1$

$n_1 + n_2 + 2N = 15$ p_2, q_2 DONE

$n_1 + n_2 + n_3 + 3N = 22$ $p_3 = 1, q_3 = 2$



Let $\widehat{D} = \varprojlim\{\widehat{D}_\omega, \sigma\}_{n=1}^{\infty}$ be an inverse limit of universal dendrites.

Since σ has the specification property, the shift homeomorphism $\widehat{\sigma}(\langle x_i\rangle_{i=1}^{\infty}) = \langle x_i\rangle_{i=2}^{\infty}$ has the specification property. (Banič, Erceg, Kennedy, and Jelić)

Since dendrites are tree-like, $\widehat{D}$ is tree-like.

In fact, $\widehat{D}$ is hereditarily decomposable,

But that is another story.

However, we do have the following questions:

Is there a fan that admits a homeomorphism with the specification property? In particular, does either the Cantor Fan or the Lelek Fan admit a homeomorphism with the specification property?

If $f : X \longrightarrow X$ is positively continuum-wise fully expansive, must it have the specification property?

Thank You!