On topological properties connected with some nonlinear operators in spaces of almost periodic functions

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Almost periodic functions in the sense of Bohr

Definition

A set $E \subseteq \mathbb{R}$ is said to be relatively dense in the sense of Bohr if there exists a number $l > 0$ such that in every open interval of the lenth l, contained in $\mathbb{R}$, there exists at least one element of E.

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Definition

A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be almost periodic in the sense of Bohr if for every $\varepsilon > 0$ the set $\{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}}|f(t+\tau) - f(t)| \leqslant \varepsilon\}$ is relatively dense in the sense of Bohr.

Almost periodic functions in the sense of Bohr

Theorem (Bochner's Criterion)

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is almost periodic in the sense of Bohr if and only if for every sequence $(\tau_n)_{n\in\mathbb{N}}$, there exists a subsequence $(\tau_{n_k})_{k\in\mathbb{N}}$ such that $(f(t+\tau_{n_k}))_{k\in\mathbb{N}}$ is uniformly convergent on $\mathbb{R}$.

Stepanov almost periodic functions

Let $\mathcal{L}$ be the σ-algebra of subsets of $\mathbb{R}$ which are measurable in the Lebesgue sense, μ – the Lebesgue measure on $\mathcal{L}$ and let $L^0(\mathbb{R})$ be the space of all $\mathcal{L}$-measurable functions $f : \mathbb{R} \to \mathbb{R}$.

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By $L^p_{loc}(\mathbb{R})$ we will denote the set of all functions $\mathbb{R} \to \mathbb{R}$ measurable in the Lebesgue sense, p-th power of absolute value of which is integrable in the Lebesgue sense over every bounded subset of $\mathbb{R}$.

Stepanov almost periodic functions

For $x, y \in L^p_{loc}(\mathbb{R})$, let us define the following quantity:

$$D_{S^p}(x,y) = \sup_{u\in\mathbb{R}} \left(\int_u^{u+1} |x(t)-y(t)|^p dt\right)^{\frac{1}{p}}.$$

It is easily to check that D_{S^p} defines a metric on the set

$$\{x \in L^p_{loc} \colon \sup_{u\in\mathbb{R}} \int_u^{u+1} |x(t)|^p dt < \infty\}.$$

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Definition

A function $x \in L^p_{loc}(\mathbb{R})$, $p \geqslant 1$, is said to be S^p-almost periodic (briefly: S^p-a.p.), if for every $\epsilon > 0$, the set

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is relatively dense in the sense of Bohr.

In the case of S^1-a.p. functions we will simply use notation: S-a.p. and we will denote the space of such functions by $S(\mathbb{R})$.

μ-almost periodic functions

Definition

A function $f \in L^0(\mathbb{R})$ is said to be μ-almost periodic if for every $\varepsilon > 0$ and $\eta > 0$, the set

$$\Big\{\tau \in \mathbb{R} : \sup_{u \in \mathbb{R}} \mu\big(\{t \in [u, u+1] : |f(t+\tau) - f(t)| \geqslant \eta\}\big) \leqslant \varepsilon\Big\}$$

is relatively dense in the sense of Bohr. By $M(\mathbb{R})$ we will denote the set of all μ-almost periodic functions.

μ-almost periodic functions

Let us consider the function $f(t) = 2 + \cos(t) + \cos(\sqrt{2}t)$ for $t \in \mathbb{R}$.

- The function f is almost periodic in the sense of Bohr.
- The function $1/f$ is μ-almost periodic and unbounded.

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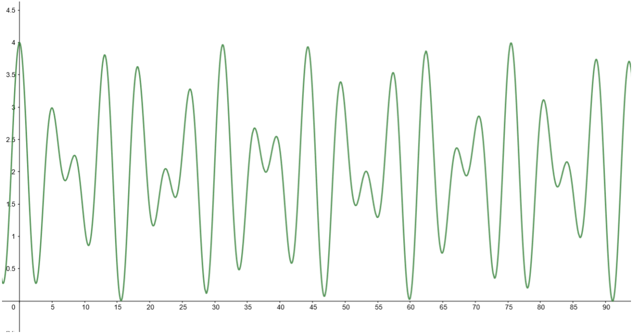

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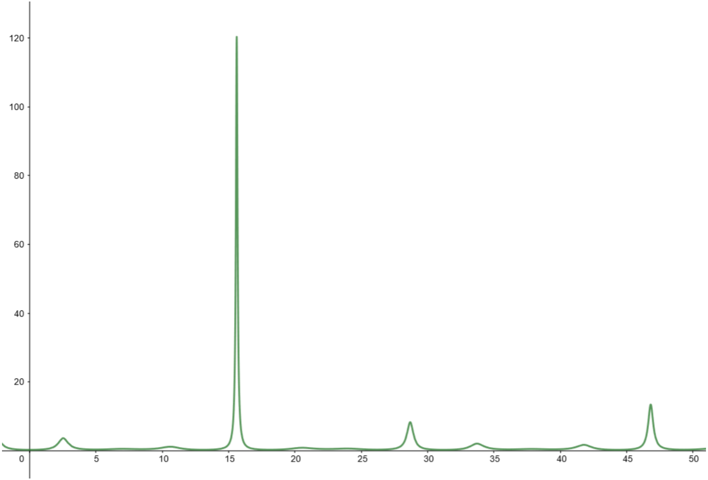

$$f(t) = \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)}$$

μ-almost periodic functions

For $\eta > 0$ and $f, g \in L^0(\mathbb{R})$ let us define

$$D(\eta; f, g) := \sup_{u \in \mathbb{R}} \mu\big(\{t \in [u, u+1] : |f(t) - g(t)| \geqslant \eta\}\big).$$

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Definition

A sequence $(f_n)_{n \in \mathbb{N}}$, where $f_n \in L^0(\mathbb{R})$ for $n \in \mathbb{N}$, is said to be D-convergent to a function $f \in L^0(\mathbb{R})$ if the following condition is satisfied:

$$\forall \varepsilon > 0 \quad \forall \eta > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad D(\eta; f_n, f) < \varepsilon.$$

The function f is said to be the D-limit of the sequence $(f_n)_{n \in \mathbb{N}}$.

Properties of μ-almost periodic functions

- Let $(\lambda_n)_{n\in\mathbb{N}}$ be any sequence of positive numbers, convergent to zero. Let us define

$$L^0_b(\mathbb{R}) = \left\{ f \in L^0(\mathbb{R}) : \lim_{n\to\infty} \sup_{u\in\mathbb{R}} \mu\left(\left\{ t \in [u, u+1] : |f(t)| \geqslant \tfrac{1}{\lambda_n} \right\}\right) = 0 \right\}.$$

If f is μ-almost periodic, then $f \in L^0_b(\mathbb{R})$.

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  If f is μ-almost periodic, then $f\in L^0_b(\mathbb{R})$.

- If f,g are μ-almost periodic functions, then $f\pm g$ and $f\cdot g$ are also μ-almost periodic functions.

Properties of μ-almost periodic functions

- Let $F\colon \Omega \to \mathbb{C}$, where $\Omega = \{t + iy \in \mathbb{C} : -a < y < a\}$, $a > 0$, be a bounded holomorphic function. Let us assume that the function $g\colon \mathbb{R} \to \mathbb{R}$ given by the formula $g(t) = F(t)$ for $t \in \mathbb{R}$, is almost periodic in the sense of Bohr. Then the function f defined by the formula

$$f(t) = \begin{cases} \dfrac{1}{g(t)} & \text{for } t \in \mathbb{R} \text{ such that } g(t) \neq 0, \\ 0 & \text{for } t \in \mathbb{R} \text{ such that } g(t) = 0, \end{cases}$$

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  is μ-almost periodic.

- If a sequence $(f_n)_{n\in\mathbb{N}}$ of μ-almost periodic functions is D-convergent to a function $f \in L^0(\mathbb{R})$, then $f \in M(\mathbb{R})$.

The space of μ-almost periodic functions

Definition

The functional $\pmb{|}\cdot\pmb{|} : L^0(\mathbb{R}) \to \mathbb{R}_+$ is defined by the formula

$$\pmb{|}f\pmb{|} = \sup_{u \in \mathbb{R}} \int_u^{u+1} \frac{|f(t)|}{1+|f(t)|} \mathrm{d}t, \quad \text{where } f \in L^0(\mathbb{R}).$$

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Using the above functional one can define, in a classical way, the metric on $L^0(\mathbb{R})$ which restricted to $L^0_b(\mathbb{R})$ is complete. Moreover, one can prove that a sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n \in L^0(\mathbb{R})$ for $n \in \mathbb{N}$, is D-convergent to a function $f \in L^0(\mathbb{R})$ if and only if $(f_n)_{n\in\mathbb{N}}$ is convergent to f in view of the metric generated by that functional.

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The space $(M(\mathbb{R}), |\!|\cdot|\!|)$ is a closed subspace of a complete space $(L^0_b(\mathbb{R}), |\!|\cdot|\!|)$, what obviously implies that it is a complete space.

Autonomous superposition operators in the space of μ-a.p. functions

Let $f\colon \mathbb{R} \to \mathbb{R}$ and let F denotes the autonomous superposition operator defined for any function $x\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$F(x)(t) = f(x(t)),$$

where $t \in \mathbb{R}$.

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Theorem

$F(L_b^0(\mathbb{R})) \subset L_b^0(\mathbb{R})$ if and only if f is a locally bounded function.

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Theorem

F is a continuous on $L_b^0(\mathbb{R})$ if and only if it is generated by a continuous function f.

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Theorem

The operator F generated by a function f is a bijection on $M(\mathbb{R})$ if and only if the function $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism.

Autonomous superposition operators in the space of μ-a.p. functions

For a continuous function $f\colon \mathbb{R} \to \mathbb{R}$, let us define

$$S(f) := \{x \in M(\mathbb{R})\colon f \circ x \text{ is S-a.p.}\}$$

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For any unbounded continuous function f, it holds $S(f) \neq M(\mathbb{R})$.

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Theorem

For any μ-a.p. function x, there exists a homeomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ x$ is S-a.p.

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Theorem

For any μ-a.p. function x, there exist an S-a.p. function y and a continuous function $z\colon \mathbb{R} \to \mathbb{R}$ such that $x = z \circ y$.

Convolution of μ-almost periodic functions

Definition

Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable in the Lebesgue sense. Define

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s)g(s)ds,$$

provided the above integral in the Lebesgue sense exists.

Convolution of μ-almost periodic functions

Remark

Convolution of a μ-almost periodic function with a function integrable in the Lebesgue sense may not exist.

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Remark

The existence of convolution of a μ-almost periodic function with a function integrable in the Lebesgue sense does not have to imply that it is a μ-almost periodic function.

Convolution of μ-almost periodic functions

Definition

The function $g_\lambda \colon \mathbb{R} \to \mathbb{R}$, where $\lambda < 0$, is defined by the formula

$$g_\lambda(t) = \begin{cases} e^{\lambda t} & \text{for} \quad t \geqslant 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

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Remark

For any function $f : \mathbb{R} \to \mathbb{R}$ locally integrable in the Lebesgue sense we have

$$(f * g_\lambda)(t) = \int_{-\infty}^{+\infty} f(s) g_\lambda(t-s) ds = \int_{-\infty}^{t} f(s) e^{\lambda(t-s)} ds = e^{\lambda t} \int_{-\infty}^{t} f(s) e^{-\lambda s} ds.$$

Convolution of μ-almost periodic functions

Theorem

If a μ-almost periodic function f satisfies the condition

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u \in \mathbb{R} \quad \forall A \subseteq [u, u+1]$$
$$\mu(A) \leqslant \delta \Longrightarrow \int_A |f(s)| \mathrm{d}s \leqslant \varepsilon,$$

then the convolution $f * g_\lambda$ exists for every $t \in \mathbb{R}$ and it is an almost periodic function in the sense of Bohr.

Convolution of μ-almost periodic functions

Theorem

Let f be a nonnegative μ-almost periodic function. If the convolution $f * g_\lambda$ exists and

$$\sup_{u \in \mathbb{R}} \int_u^{u+1} f(s)\mathrm{d}s = +\infty,$$

then it is not a μ-almost periodic function.

Convolution of μ-almost periodic functions

Theorem

For every $\lambda < 0$ it holds

$$\lim_{t\to+\infty} \frac{e^{\lambda t}}{2+\cos(t)+\cos(\sqrt{2}t)} = 0.$$

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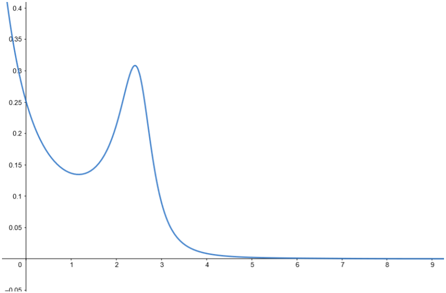

$$f(t) = \frac{e^{-t}}{2+\cos(t)+\cos(\sqrt{2}t)}$$

Convolution of μ-almost periodic functions

Theorem

For every function $f : \mathbb{R} \to (0, +\infty)$, every $a \in \mathbb{R}$ and every $\varepsilon > 0$ there exist $\alpha \in \mathbb{R}$ such that

$$|a - \alpha| < \varepsilon \quad \text{and} \quad \limsup_{t \to +\infty} \frac{f(t)}{2 + \cos(t) + \cos(\alpha t)} = +\infty.$$

Convolution of μ-almost periodic functions

Theorem

The set $\bigcup_{\lambda<0} S_\lambda$, where

$$S_\lambda = \{\alpha \in \mathbb{R} \setminus \mathbb{Q}\colon \frac{1}{2+\cos(\cdot)+\cos(\alpha\cdot)} * g_\lambda \quad \text{exists}\},$$

is of the first Baire category. Moreover, $\bigcap_{\lambda<0} S_\lambda$ and S_{λ_0}, for $\lambda_0 < 0$, are of the first Baire category. Thereby $\bigcap_{\lambda<0} S'_\lambda$, S'_{λ_0}, for $\lambda_0 < 0$, and $\bigcup_{\lambda<0} S'_\lambda$, where $S'_\lambda = \mathbb{R} \setminus S_\lambda$, are of the second Baire category.

Convolution of μ-almost periodic functions

Theorem

For every $a \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $\lambda < 0$

$$|a - \alpha| < \epsilon \quad \text{and} \quad \int_{-\infty}^{0} \frac{e^{-\lambda t}}{2 + \cos t + \cos(\alpha t)} \mathrm{d}t = +\infty.$$

In other words, the set

$$\bigcap_{\lambda<0} S'_\lambda = \left\{\alpha \in \mathbb{R} \setminus \mathbb{Q} \colon \frac{1}{2 + \cos \cdot + \cos(\alpha \cdot)} * g_\lambda \quad \text{does not exist for all } \lambda < 0\right\}$$

is dense in $\mathbb{R}$. Thereby for $\lambda_0 < 0$ the sets S'_{λ_0} and $\bigcup_{\lambda<0} S'_\lambda$ are also dense in $\mathbb{R}$.

Convolution of μ-almost periodic functions

Theorem

$\bigcap_{\lambda<0} S_\lambda$ is a set with the cardinality of the continuum. Moreover, the set $\bigcap_{\lambda<0} S'_\lambda$ is also with the cardinality of the continuum. Thereby $\bigcup_{\lambda<0} S_\lambda$ and $\bigcup_{\lambda<0} S'_\lambda$ are also with the cardinality of the continuum.

Applications

Let us consider the first order linear differential equation of the form

$$x'(t) = \lambda x(t) + f(t), \qquad t \in \mathbb{R},$$

where $\lambda < 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous μ-almost periodic function.

Applications

Let us consider the first order linear differential equation of the form

$$x'(t) = \lambda x(t) + f(t), \qquad t \in \mathbb{R},$$

where $\lambda < 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous μ-almost periodic function.

By (C) we denote the following function

$$x(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s) \mathrm{d}s, \qquad t \in \mathbb{R}.$$

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where $\lambda < 0$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous μ-almost periodic function.

Theorem

Under the above asumptions one of the following claims holds:

- the function (C) is μ-almost periodic solution to the above equation;
- the function (C) is a solution to the above equation, however it is not μ-almost periodic;
- the function (C) is not a solution to the above equation.

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Thank you

for

attention