A study of a new dimension for finite lattices

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Abstract

In this talk we give a new notion of dimension, called **dimension Dind**, in the area of finite lattices. We present:

- many properties of Dind and
- relations between Dind and known dimensions of finite lattices; the covering dimension and the Krull dimension.

We begin this talk with some useful preliminary notes.

Partially Ordered Sets

A binary relation $R \subseteq L \times L$ on a non-empty set L is called **partial order relation** if it satisfies the following:

- $(a, a) \in R$, for any $a \in L$.
- If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$, for any $a, b \in L$.
- If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for any $a, b, c \in L$.

The pair (L, R) is called **partially ordered set** or **poset**. In what follows we use the symbol aRb instead of $(a, b) \in R$ and also we use the notation $\leqslant$ instead of R.
Sometimes we refer to a poset L without stating the symbol $\leqslant$.
Finally, for a poset $(L, \leqslant)$ we write $a < b$ if $a \leqslant b$ and $a \neq b$.

Supremum and Infimum

Let $(L, \leqslant)$ be a poset and $A \subseteq L$.

- An element $x \in L$ is called **upper bound** of A if $a \leqslant x$ for every $a \in A$. The least upper bound of A, if this exists, is called **supremum** of A and is denoted by $\sup(A)$ or $\bigvee A$. Especially, if $x, y \in A$, we write $x \vee y$ instead of $\sup\{x, y\}$.
- An element $x \in L$ is called **lower bound** of A if $x \leqslant a$ for every $a \in A$. The greatest lower bound of A, if this exists, is called **infimum** of A and is denoted by $\inf(A)$ or $\bigwedge A$. Especially, if $x, y \in A$, we write $x \wedge y$ instead of $\inf\{x, y\}$.

Isomorphism between posets

Let $(L, \leqslant_L)$ and $(M, \leqslant_M)$ be two posets. An 1-1 function $f : L \to M$ is called **isomorphism** if for every $x_1, x_2 \in L$, we have:

$$x_1 \leqslant_L x_2 \Longleftrightarrow f(x_1) \leqslant_M f(x_2).$$

Lattices-Finite Lattices

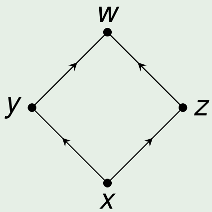
- A poset $(L, \leqslant)$ is called **lattice** if every finite subset of L has supremum and infimum in L.
- A poset $(L, \leqslant)$ is called **finite lattice** if the set L is finite.

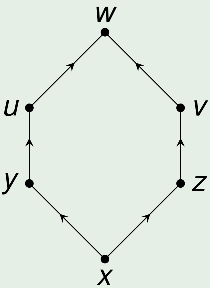
Clearly, every finite lattice has the minimum and the maximum element. In our talk, we denote by 0_L and 1_L the bottom and the top element of L, respectively.

Hasse Diagrams

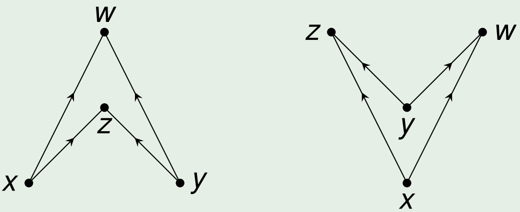
Usually, a finite poset or a finite lattice $(L, \leqslant)$ is represented through diagrams where for any two elements $x, y \in L$ such that $x < y$ we draw the following arrow:

Examples of lattices with Hasse diagrams





Examples of non-lattices with Hasse diagrams



Notations

Let $(L, \leqslant)$ be a finite lattice.

- For any element x of L, we denote by
  - $\uparrow x = \{y \in L : x \leqslant y\}$
  - $\downarrow x = \{y \in L : y \leqslant x\}$ and
  - $\downarrow^* x = (\downarrow x) \setminus \{0_L\}$.
- For any element x of L, we denote by

$$x^* = \max\{y \in L : y \wedge x = 0_L\},$$

  called the **pseudocomplement** of x.

We mention that for any element $x \in L$, the set $\uparrow(x^* \vee x)$ is a lattice.

Covers of lattices

Let $(L, \leqslant)$ be a finite lattice.

- A subset V of L is called a **cover** of L if $0_L \notin V$ and $\bigvee V = 1_L$.
- A subset U of a lattice L is called a **refinement** of a cover V of L, writing $U \succ V$, if for each $u \in U$, there exists $v \in V$ such that $u \leqslant v$.
- A cover U of a lattice L is called a **minimal** if $U \subseteq V$ for every cover V of L which is a refinement of U.
- A subset A of L is said to be a **set of pairwise disjoint elements** if $0_L \notin A$ and for every $x, y \in A$, with $x \neq y$, we have $x \wedge y = 0_L$.

Now, we insert the notion of dimension Dind in the class of finite lattices and present basic properties of this dimension.

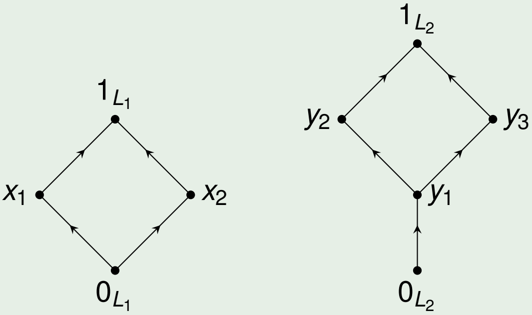
Dimension Dind

Let $(L, \leqslant)$ be a finite lattice. The **dimension Dind** of L is defined as follows:

- $\mathrm{Dind}(L) = -1$ if and only if $L = \{0_L\}$.
- $\mathrm{Dind}(L) \leqslant k$, where $k \in \{0, 1, 2, \ldots\}$, if for every finite cover V of L, there exists a finite subset U of L, which is a set of pairwise disjoint elements, $U \succ V$ and $\mathrm{Dind}(\uparrow(u^* \vee u)) \leqslant k - 1$, for every $u \in L \setminus \downarrow^*(\bigvee U)$.
- $\mathrm{Dind}(L) = k$, where $k \in \{0, 1, 2, \ldots\}$, if $\mathrm{Dind}(L) \leqslant k$ and $\mathrm{Dind}(L) \nleqslant k - 1$.

Examples

We consider the finite lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ represented by the following diagrams:



For the above lattices we have $\mathrm{Dind}(L_1) = 0$ and $\mathrm{Dind}(L_2) = 1$.

However, in our study, we prove that we can always construct a finite lattice L with dimension Dind being any natural number k.

Theorem

For any $k \in \{1, 2, \ldots\}$, there exists a finite lattice L with $\mathrm{Dind}(L) = k$.

We continue the study of Dind presenting more results.

Proposition 1

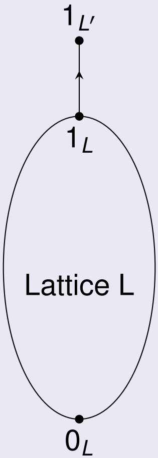
Isomorphic lattices have the same dimension Dind.

Proposition 2

Let $(L, \leqslant)$ be a finite lattice and $k \in \mathbb{N}$. Then, $\mathrm{Dind}(L) \leqslant k$ if and only if for every minimal cover V of L, there exists a subset U of L, which is a set of pairwise disjoint elements, $U \succ V$ and $\mathrm{Dind}(\uparrow(u \vee u^*)) \leqslant k - 1$, for every $u \in L \smallsetminus \downarrow^*(\bigvee U)$.

Proposition 3

Let $(L, \leqslant)$ be a finite lattice and $L' = L \cup \{1_{L'}\}$ be the finite lattice of the following diagram. Then $\mathrm{Dind}(L') = 0$.



Now, we present properties of the dimension Dind of different finite lattices. Especially, we present the dimension Dind of the sum and products of finite lattices.

Linear sum of lattices

The **linear sum** $(L_1 \oplus L_2, \leqslant)$ of two lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$, where $L_1 \cap L_2 = \varnothing$, is the lattice $(L_1 \cup L_2, \leqslant)$, where the relation $\leqslant$ is defined as follows:

$$x \leqslant y \Leftrightarrow \begin{cases} x, y \in L_1 \text{ and } x \leqslant_1 y \\ x, y \in L_2 \text{ and } x \leqslant_2 y \\ x \in L_1,\ y \in L_2. \end{cases}$$

Proposition 4

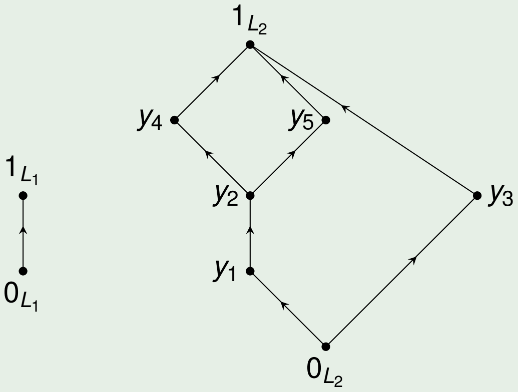
In general, the relation

$$\mathrm{Dind}(L_1 \oplus L_2) \leqslant \mathrm{Dind}(L_1) + \mathrm{Dind}(L_2)$$

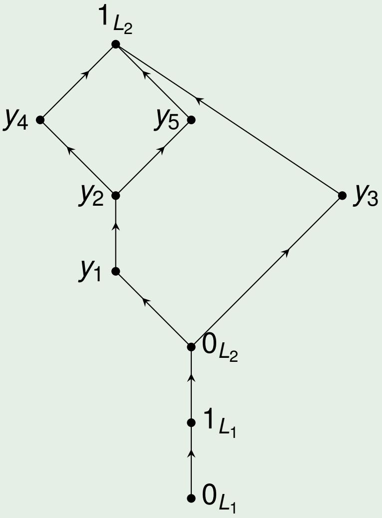
does not hold.

Example

We consider the following lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$:



The linear sum $L_1 \oplus L_2$ is given in the following diagram:



Sum and product properties for Dind

We observe that $\mathrm{Dind}(L_1) = \mathrm{Dind}(L_2) = 0$ and $\mathrm{Dind}(L_1 \oplus L_2) = 1$. Thus,

$$\mathrm{Dind}(L_1 \oplus L_2) \nleq \mathrm{Dind}(L_1) + \mathrm{Dind}(L_2).$$

Cartesian product of lattices

The **Cartesian product** of two lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ is the lattice $(L_1 \times L_2, \leqslant)$, where

$$L_1 \times L_2 = \{(x, y) : x \in L_1 \text{ and } y \in L_2\}$$

and the relation $\leqslant$ is defined as follows:

$$(x_1, y_1) \leqslant (x_2, y_2) \Leftrightarrow x_1 \leqslant_1 x_2 \text{ and } y_1 \leqslant_2 y_2.$$

Proposition 5

For any two finite lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ the following relations hold:

1. $\mathrm{Dind}(L_1) \leqslant \mathrm{Dind}(L_1 \times L_2)$ and
2. $\mathrm{Dind}(L_2) \leqslant \mathrm{Dind}(L_1 \times L_2)$.

Lexicographic product of lattices

For two lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ the **lexicographic product** $L_1 \diamond L_2$ is the lattice $(L_1 \times L_2, \leqslant)$, where the relation $\leqslant$ is defined as follows:

$$(x_1, y_1) \leqslant (x_2, y_2) \Leftrightarrow \begin{cases} x_1 <_1 x_2 \text{ or} \\ x_1 = x_2 \text{ and } y_1 \leqslant_2 y_2. \end{cases}$$

Sum and product properties for Dind

Proposition 6

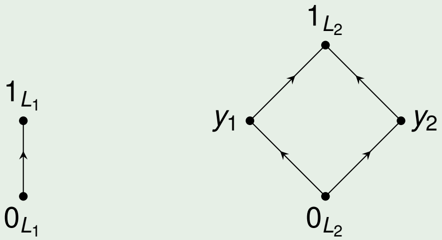
In general, the relation

$$\mathrm{Dind}(L_1 \diamond L_2) \leqslant \mathrm{Dind}(L_1) + \mathrm{Dind}(L_2)$$

does not hold.

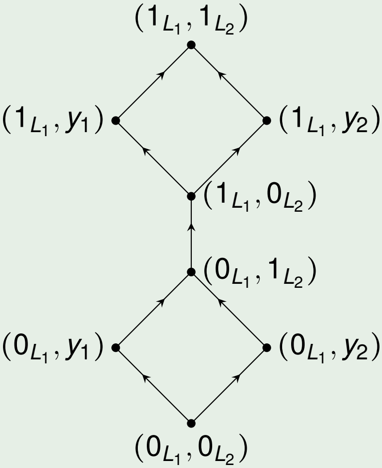
Example

We consider the following lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$:



Sum and product properties for Dind

The lexicographic product $L_1 \diamond L_2$ is given in the following diagram:



Sum and product properties for Dind

We observe that $\mathrm{Dind}(L_1) = \mathrm{Dind}(L_2) = 0$ and $\mathrm{Dind}(L_1 \diamond L_2) = 1$. Thus,

$$\mathrm{Dind}(L_1 \diamond L_2) \nleq \mathrm{Dind}(L_1) + \mathrm{Dind}(L_2).$$

Proposition 7

In general the relations:

1. $\mathrm{Dind}(L_1 \diamond L_2) \leqslant \mathrm{Dind}(L_1 \times L_2)$
2. $\mathrm{Dind}(L_1 \times L_2) \leqslant \mathrm{Dind}(L_1 \diamond L_2)$

do not hold, that is we can not compare the dimension Dind of the Cartesian product with the dimension Dind of the lexicographic product.

Rectangular product of lattices

The **rectangular product** of two finite lattices $(L_1, \leqslant_1)$ and $(L_2, \leqslant_2)$ is the lattice $(L_1 \square L_2, \leqslant)$, where

$$L_1 \square L_2 = \{(x, y) \in L_1 \times L_2 : x \neq 0_{L_1} \text{ and } y \neq 0_{L_2}\} \cup \{(0_{L_1}, 0_{L_2})\}$$

and the relation $\leqslant$ is defined as follows:

$$(x_1, y_1) \leqslant (x_2, y_2) \Leftrightarrow x_1 \leqslant_1 x_2 \text{ and } y_1 \leqslant_2 y_2.$$

Proposition 8

In general the relations:

1. $\mathrm{Dind}(L_1 \square L_2) \leqslant \mathrm{Dind}(L_1 \times L_2)$
2. $\mathrm{Dind}(L_1 \times L_2) \leqslant \mathrm{Dind}(L_1 \square L_2)$

do not hold, that is we can not compare the dimension Dind of the Cartesian product with the dimension Dind of the rectangular product.

Proposition 9

In general the relations:

1. $\mathrm{Dind}(L_1 \diamond L_2) \leqslant \mathrm{Dind}(L_1 \,\square\, L_2)$
2. $\mathrm{Dind}(L_1 \,\square\, L_2) \leqslant \mathrm{Dind}(L_1 \diamond L_2)$

do not hold, that is we can not compare the dimension Dind of the lexicographic product with the dimension Dind of the rectangular product.

Now, we present some additional remarks for the dimension Dind for the class of finite lattices, comparing it with the covering dimension and the Krull dimension.

The meaning of order

Let $(L, \leqslant)$ be a finite lattice. The **order** of a subset C of L, denoted by $\mathrm{ord}(C)$, is defined to be k, where $k \in \{0, 1, 2, \ldots\}$, if and only if the infimum of any $k + 2$ distinct elements of C is 0_L and there exist $k + 1$ distinct elements of C whose infimum is not 0_L.

Covering dimension

Let $(L, \leqslant)$ be a finite lattice. The **covering dimension** of L is defined as follows:

1. $\dim(L) \leqslant k$, where $k \in \{0, 1, 2, \ldots\}$, if and only if for every cover C of L, there exists a cover R of L, refinement of C with $\operatorname{ord}(R) \leqslant k$.
2. $\dim(L) = k$, where $k \in \{0, 1, 2, \ldots\}$, if $\dim(\mathrm{L}) \leqslant k$ and $\dim(L) \nleqslant k - 1$.

Proposition 10

In general, we can not compare the dimensions Dind and dim for finite lattices. That is, the relations:

1. $\mathrm{Dind}(L) \leqslant \dim(L)$
2. $\dim(L) \leqslant \mathrm{Dind}(L)$

do not hold.

Prime filters

A non-empty subset F of a lattice $(L, \leqslant)$ is called **filter** if F has the following properties:

1. $F \neq L$.
2. If $x \in F$ and $x \leqslant y$, then $y \in F$.
3. If $x, y \in F$, then $x \wedge y \in F$.

A filter F is called **prime** if for every $x, y \in L$ with $x \vee y \in F$, we have $x \in F$ or $y \in F$. The set of all prime filters of a lattice L is usually denoted by $\mathcal{PF}(L)$.

Krull dimension

If $\mathcal{PF}(L) \neq \varnothing$, then the **Krull dimension** of $(L, \leqslant)$ is defined as follows:

$$\mathrm{Kdim}(L) = \sup\{k : \text{there exist prime filters } F_0 \subset F_1 \subset \cdots \subset F_k\}.$$

Proposition 11

In general, we can not compare the dimensions Dind and Kdim for finite lattices. That is, the relations:

1. $\mathrm{Dind}(L) \leqslant \mathrm{Kdim}(L)$
2. $\mathrm{Kdim}(L) \leqslant \mathrm{Dind}(L)$

do not hold.

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Thank You!!!