A small Boolean algebra that is Grothendieck but not Nikodym

Damian Głodkowski

joint work with Agnieszka Widz

Institute of Mathematics
Polish Academy of Sciences

SUMTOPO 2024

The Grothendieck property

Definition

A Banach space X has the **Grothendieck property**, if every weak*-convergent sequence in X^* is weakly convergent.

The Grothendieck property

Definition

A Banach space X has the **Grothendieck property**, if every weak*-convergent sequence in X^* is weakly convergent.

Grothendieck: if $\mathbb{B}$ is a complete Boolean algebra and $\mathrm{St}(\mathbb{B})$ is its Stone space, then $C(\mathrm{St}(\mathbb{B}))$ has the Grothendieck property.
In particular, $\ell_\infty \equiv C(\mathrm{St}(\mathcal{P}(\mathbb{N})))$ has the Grothendieck property.

The Grothendieck property

Definition

A Banach space X has the **Grothendieck property**, if every weak*-convergent sequence in X^* is weakly convergent.

Grothendieck: if $\mathbb{B}$ is a complete Boolean algebra and $\mathrm{St}(\mathbb{B})$ is its Stone space, then $C(\mathrm{St}(\mathbb{B}))$ has the Grothendieck property.
In particular, $\ell_\infty \equiv C(\mathrm{St}(\mathcal{P}(\mathbb{N})))$ has the Grothendieck property.

Definition

A Boolean algebra $\mathbb{B}$ has the **Grothendieck property**, if $C(\mathrm{St}(\mathbb{B}))$ has the Grothendieck property.

Measures on Boolean algebras

A measure on $\mathbb{B}$ = a finitely additive real-valued bounded function on $\mathbb{B}$

Measures on Boolean algebras

A measure on $\mathbb{B}$ = a finitely additive real-valued bounded function on $\mathbb{B}$

Folklore

- Every measure on $\mathbb{B}$ uniquely extends to a Radon measure on $\mathrm{St}(\mathbb{B})$
- The restriction of a Radon measure on $\mathrm{St}(\mathbb{B})$ to the clopen sets is a measure on $\mathbb{B}$

The Nikodym property

We will say that a sequence $(\nu)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ is **pointwise convergent** if there exists a measure ν on $\mathbb{B}$ such that for all $A \in \mathbb{B}$ we have $\nu_n(A) \to \nu(A)$.

Definition

We say that a Boolean algebra $\mathbb{B}$ has the **Nikodym property**, if every pointwise convergent sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ is bounded in norm (i.e. $\sup_{n\in\mathbb{N}} \|\nu_n\| < \infty$).

Examples

- σ-complete algebras have both the Nikodym and Grothendieck properties
- countable algebras have none of these properties

Examples

- σ-complete algebras have both the Nikodym and Grothendieck properties
- countable algebras have none of these properties
- Schachermayer (1982): the Boolean algebra consisting of Jordan measurable subsets of $[0, 1]$ has the Nikodym property, but not the Grothendieck property

Examples

- σ-complete algebras have both the Nikodym and Grothendieck properties
- countable algebras have none of these properties
- Schachermayer (1982): the Boolean algebra consisting of Jordan measurable subsets of $[0, 1]$ has the Nikodym property, but not the Grothendieck property
- Talagrand (1984): Assuming CH there is a Boolean algebra with the Grothendieck property and without the Nikodym property

Examples

- σ-complete algebras have both the Nikodym and Grothendieck properties
- countable algebras have none of these properties
- Schachermayer (1982): the Boolean algebra consisting of Jordan measurable subsets of $[0, 1]$ has the Nikodym property, but not the Grothendieck property
- Talagrand (1984): Assuming CH there is a Boolean algebra with the Grothendieck property and without the Nikodym property

Open question: Is there a Boolean algebra with the Grothendieck property and without the Nikodym property in ZFC?

The main result

Theorem (G. & Widz)

There is a σ-centered (and so ccc) notion of forcing $\mathbb{P}$ such that

$$\mathbb{P} \Vdash \quad \begin{array}{l}\text{there exists a Boolean algebra of cardinality } \omega_1 \text{ with the} \\ \text{Grothendieck property and without the Nikodym property}\end{array}$$

In particular, the existence of such an algebra is consistent with $\neg$CH.

The main result

Theorem (G. & Widz)

There is a σ-centered (and so ccc) notion of forcing $\mathbb{P}$ such that

$$\mathbb{P} \Vdash \quad \text{there exists a Boolean algebra of cardinality } \omega_1 \text{ with the Grothendieck property and without the Nikodym property}$$

In particular, the existence of such an algebra is consistent with $\neg$CH.

This algebra consists of Borel subsets of the Cantor set.

Normal sequences

We say that a sequence $(\nu)_{n\in\mathbb{N}}$ of measures on a Boolean algebra $\mathbb{B}$ is **normal** if

- $\forall n \in \mathbb{N}\ \|\nu_n\| = 1$,
- the Radon measures $\widetilde{\nu}_n$ on $\mathrm{St}(\mathbb{B})$ extending ν_n are concentrated on pairwise disjoint Borel sets.

Normal sequences

We say that a sequence $(\nu)_{n\in\mathbb{N}}$ of measures on a Boolean algebra $\mathbb{B}$ is **normal** if

- $\forall n \in \mathbb{N}\ \|\nu_n\| = 1$,
- the Radon measures $\widetilde{\nu}_n$ on $\mathrm{St}(\mathbb{B})$ extending ν_n are concentrated on pairwise disjoint Borel sets.

Fact

If $\mathbb{B}$ does not have the Grothendieck property, then there is a normal sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{B}$ such that $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ converges in the weak*-topology, but not weakly.

Property $(\mathcal{G})$

We say that a Boolean algebra $\mathbb{B}$ satisfies property $(\mathcal{G})$, if for every normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ there is $G \in \mathbb{B}$ and pairwise disjoint sets $(H_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ such that

- For infinitely many $n \in \mathbb{N}$
  - $|\nu_n(G \cap H_n)| \geqslant 0.3$ and
  - $|\nu_n|(H_n) \geqslant 0.9$.
- For infinitely many $n \in \mathbb{N}$
  - $G \cap H_n = \varnothing$ and
  - $|\nu_n|(H_n) \geqslant 0.9$.

Property $(\mathcal{G})$

We say that a Boolean algebra $\mathbb{B}$ satisfies property $(\mathcal{G})$, if for every normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ there is $G \in \mathbb{B}$ and pairwise disjoint sets $(H_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ such that

- For infinitely many $n \in \mathbb{N}$
  - $|\nu_n(G \cap H_n)| \geqslant 0.3$ and
  - $|\nu_n|(H_n) \geqslant 0.9$.
- For infinitely many $n \in \mathbb{N}$
  - $G \cap H_n = \varnothing$ and
  - $|\nu_n|(H_n) \geqslant 0.9$.

Proposition

If $\mathbb{B}$ satisfies $(\mathcal{G})$, then it has the Grothendieck property.

Balanced sets

Notation:

- the Cantor set: $C = \{-1,1\}^\omega$
- $\langle s \rangle = \{x \in C : x \restriction m = s\}$ for $s \in \{-1,1\}^m$
- λ is the measure on $\mathrm{Bor}(C)$ such that $\lambda(\langle s \rangle) = 1/2^m$ for $s \in \{-1,1\}^m$

Balanced sets

Let $m \in \mathbb{N}$ and $\varepsilon > 0$. We say that $A \in \mathrm{Bor}(C)$ is (m, ε)-**balanced**, if for every $s \in \{-1, 1\}^m$ we have

$$\lambda(A \cap \langle s \rangle) < \frac{\varepsilon}{2^m m} \text{ or } \lambda(\langle s \rangle \setminus A) < \frac{\varepsilon}{2^m m},$$

Balanced sets

Let $m \in \mathbb{N}$ and $\varepsilon > 0$. We say that $A \in \mathrm{Bor}(C)$ is (m, ε)-**balanced**, if for every $s \in \{-1, 1\}^m$ we have

$$\lambda(A \cap \langle s \rangle) < \frac{\varepsilon}{2^m m} \text{ or } \lambda(\langle s \rangle \backslash A) < \frac{\varepsilon}{2^m m},$$

and for every $s \in \{-1, 1\}^m$ and $r > m$

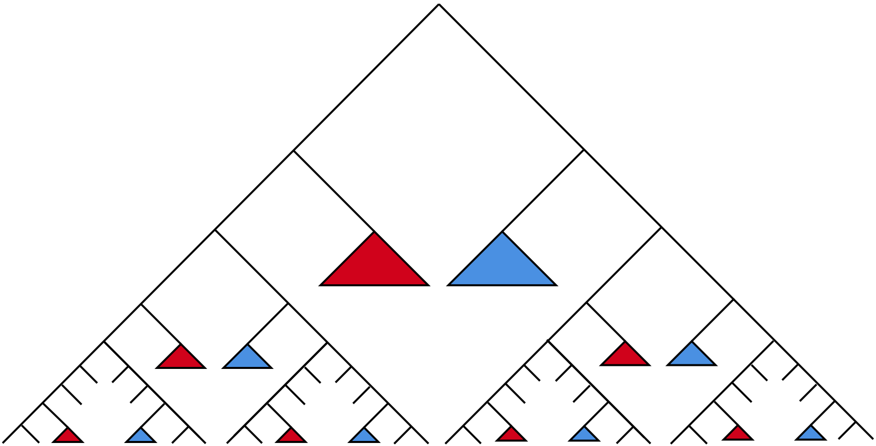
$$\left| \int_{A \cap \langle s \rangle} \delta_r d\lambda \right| < \frac{\varepsilon}{2^m r},$$

where

$$\delta_r(x) = x(r)$$

Example

The union of red and blue triangles is $(2^n, 2^{n+2}/2^{2^n})$-balanced for $n \in \mathbb{N}$



A Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is **balanced** if for every finite family $\mathcal{A} \subseteq \mathbb{B}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that every $A \in \mathcal{A}$ is (m, ε)-balanced.

Balanced algebras

A Boolean algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is **balanced** if for every finite family $\mathcal{A} \subseteq \mathbb{B}$ and $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that every $A \in \mathcal{A}$ is (m, ε)-balanced.

Fact

If $\mathbb{B} \subseteq \mathrm{Bor}(C)$ is balanced, then it does not have the Nikodym property.

Proof: the sequence

$$\varphi_n(A) = n \int_A \delta_n d\lambda$$

is pointwise convergent to 0 on $\mathbb{B}$ but is not bounded in norm.

Balanced algebras

Theorem (Talagrand)

Assume CH. Then there exists a balanced Boolean algebra with the Grothendieck property.

Balanced algebras

Theorem (Talagrand)

Assume CH. Then there exists a balanced Boolean algebra with the Grothendieck property.

Theorem (G. & Widz)

- Assume CH. Then there is a balanced Boolean algebra satisfying $(\mathcal{G})$.
- It is consistent with any possible size of $\mathfrak{c}$ that there exists a balanced algebra of size ω_1 satisfying $(\mathcal{G})$.

Sketch of the construction under CH

We construct a balanced algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ with the property $(\mathcal{G})$ as a union

$$\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha,$$

where $\mathbb{B}_\alpha$ are constructed by induction.

Sketch of the construction under CH

We construct a balanced algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ with the property $(\mathcal{G})$ as a union

$$\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}_\alpha,$$

where $\mathbb{B}_\alpha$ are constructed by induction.

- We start with $\mathbb{B}_0 = \mathrm{Clop}(C)$

Sketch of the construction under CH

We construct a balanced algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ with the property $(\mathcal{G})$ as a union

$$\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha,$$

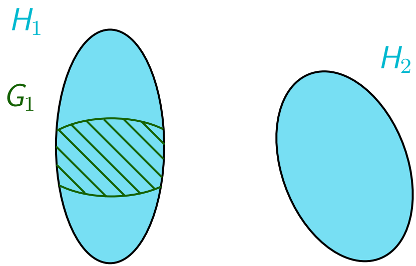
where $\mathbb{B}_\alpha$ are constructed by induction.

- We start with $\mathbb{B}_0 = \mathrm{Clop}(C)$
- If β is a limit ordinal, then

$$\mathbb{B}_\beta = \bigcup_{\alpha<\beta} \mathbb{B}_\alpha$$

- While constructing $\mathbb{B}_{\alpha+1}$ we are given some normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}_\alpha$ and we add a new set that is a witness for the property $(\mathcal{G})$ (keeping everything balanced).

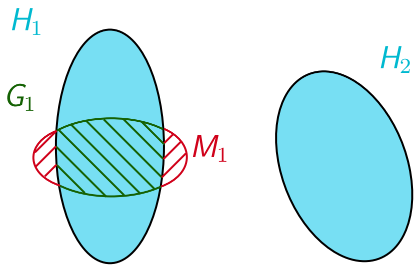
Construction of a witness for $(\mathcal{G})$



We find $n_1, n_2 \in \mathbb{N}$, disjoint sets $H_1, H_2 \in \mathbb{B}_\alpha$ and $G_1 \subseteq H_1$ such that

- $|\nu_{n_1}|(H_1), |\nu_{n_2}|(H_2) > 0.9$,
- $|\nu_{n_1}(G_1)| > 0.3$,
- other technical conditions that will allow us to continue the construction

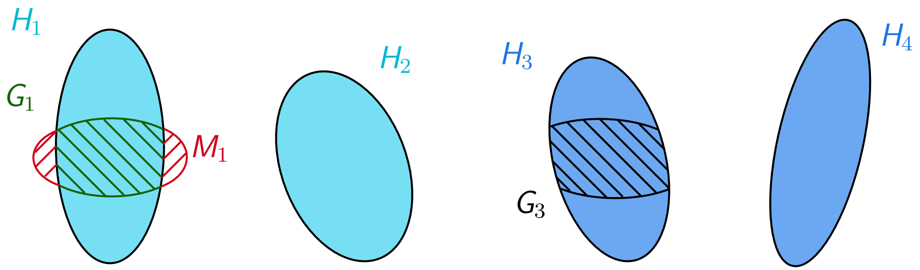
Construction of a witness for $(\mathcal{G})$



Then we find a "very small" set $M_1 \in \mathbb{B}_\alpha$ such that

- $M_1 \cap (H_1 \cup H_2) = \varnothing$
- $\langle \mathbb{A}_0 \cup \{G_1 \cup M_1\} \rangle$ is "sufficiently well balanced", where $\mathbb{A}_0$ is a finite subalgebra of $\mathbb{B}_\alpha$

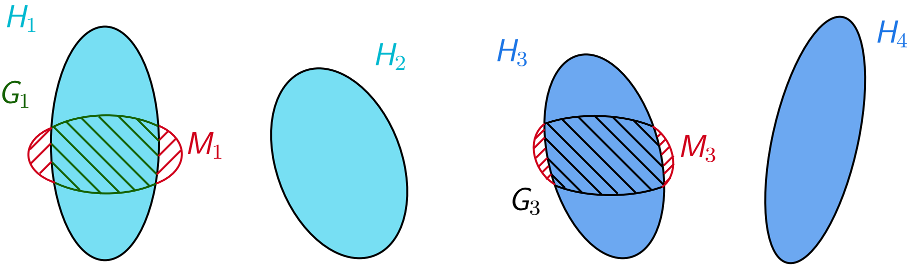
Construction of a witness for $(\mathcal{G})$



We find $n_3, n_4 \in \mathbb{N}$, disjoint sets $H_3, H_4 \in \mathbb{B}_\alpha$ and $G_3 \subseteq H_3$ such that

- $|\nu_{n_3}|(H_3), |\nu_{n_4}|(H_4) > 0.9$,
- $|\nu_{n_3}(G_1)| > 0.3$,
- other technical conditions that will allow us to continue the construction

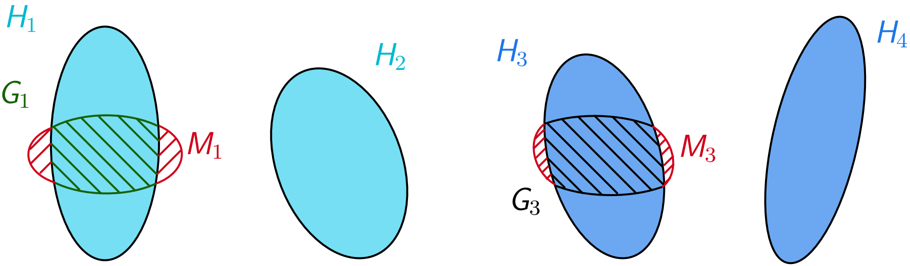
Construction of a witness for $(\mathcal{G})$



Then we find a "very small" set $M_3 \in \mathbb{B}_\alpha$ such that

- $M_3 \cap (H_1 \cup H_2 \cup H_3 \cup H_4 \cup M_1) = \varnothing$
- $\langle \mathbb{A}_1 \cup \{G_1 \cup M_1 \cup G_2 \cup M_2\}\rangle$ is "sufficiently well balanced", where $\mathbb{A}_1$ is a finite subalgebra of $\mathbb{B}_\alpha$ that is bigger than $\mathbb{A}_0$

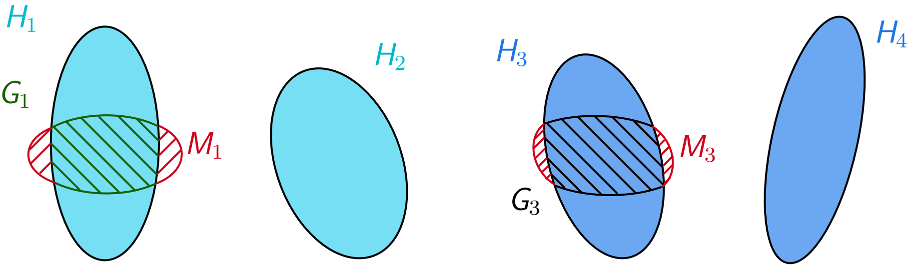
Construction of a witness for $(\mathcal{G})$



We finish taking

$$G = \bigcup_{i \in Odd} (G_i \cup M_i)$$

Construction of a witness for $(\mathcal{G})$



We finish taking

$$G = \bigcup_{i \in Odd} (G_i \cup M_i)$$

Then

- $\mathbb{B}_{\alpha+1} = \langle \mathbb{B}_\alpha \cup \{G\} \rangle$ is balanced
- G is a witness for the property $(\mathcal{G})$ for $(\nu_n)_{n \in \mathbb{N}}$

Forcing

For a countable Boolean algebra $\mathbb{B}$ we fix a representation as an increasing union of finite subalgebras:

$$\mathbb{B} = \bigcup_{n \in \mathbb{N}} = \mathbb{B}_n$$

We define a notion of forcing $\mathbb{P}$. Conditions are of the form

$$p = (k^p, (m_n^p)_{n \leqslant k^p}, (G_n^p)_{n \leqslant k^p}, (H_n^p)_{n \leqslant k^p}, \mathcal{M}^p),$$

where

- $k^p \in \mathbb{N}$,
- $(m_n^p)_{n \leqslant k^p}$ is a strictly increasing sequence of natural numbers,
- $\mathcal{M}^p$ is a finite set of probability measures on $\mathbb{B}$ such that $\lambda \upharpoonright \mathbb{B} \in \mathcal{M}^p$,
- $(G_n^p)_{n \leqslant k^p}$ and $(H_n^p)_{n \leqslant k^p}$ are sequences of elements of $\mathbb{B}$ such that
    - $G_n^p \cap G_l^p = H_n^p \cap H_l^p = G_n^p \cap H_l^p = \varnothing$ for $n \neq l$,
    - $\mu\left(\bigcup_{n \leqslant k^p} (G_n^p \cup H_n^p)\right) < 0.1$ for all $\mu \in \mathcal{M}^p$,
    - $\left\langle \mathbb{B}_n \cup \{\bigcup_{i \leqslant k^p} G_i^p\} \right\rangle$ is $(m_n, 2^{-n})-$balanced for $n \leqslant k^p$.

Forcing

For a countable Boolean algebra $\mathbb{B}$ we fix a representation as an increasing union of finite subalgebras:

$$\mathbb{B} = \bigcup_{n\in\mathbb{N}} = \mathbb{B}_n$$

We define a notion of forcing $\mathbb{P}$. Conditions are of the form

$$p = (k^p, (m_n^p)_{n\leqslant k^p}, (G_n^p)_{n\leqslant k^p}, (H_n^p)_{n\leqslant k^p}, \mathcal{M}^p),$$

where $q \leqslant p$, if

- $k^q \geqslant k^p$,
- $m_n^q = m_n^p$ for $n \leqslant k^p$,
- $G_n^q = G_n^p$ for $n \leqslant k^p$,
- $H_n^q = H_n^p$ for $n \leqslant k^p$,
- $\mathcal{M}^q \supseteq \mathcal{M}^p$.

Properties of $\mathbb{P}$

Let $\mathbb{G}$ be $\mathbb{P}$-generic over V. In $V[\mathbb{G}]$ we define

$$G = \bigcup \{ G_n^p : p \in \mathbb{G}, n \leqslant k^p \}$$

Properties of $\mathbb{P}$

Let $\mathbb{G}$ be $\mathbb{P}$-generic over V. In $V[\mathbb{G}]$ we define

$$G = \bigcup \{ G_n^p : p \in \mathbb{G}, n \leqslant k^p \}$$

Then

- the algebra $\langle \mathbb{B} \cup \{ G \} \rangle$ is balanced,

Properties of $\mathbb{P}$

Let $\mathbb{G}$ be $\mathbb{P}$-generic over V. In $V[\mathbb{G}]$ we define

$$G = \bigcup \{ G_n^p : p \in \mathbb{G}, n \leqslant k^p \}$$

Then

- the algebra $\langle \mathbb{B} \cup \{ G \} \rangle$ is balanced,
- if $(\nu_n)_{n\in\mathbb{N}}$ is a normal sequence such that $(|\nu_n|)_{n\in\mathbb{N}}$ converges to a measure $\nu \in \mathcal{M}^p$ for some $p \in \mathbb{G}$, then G is a witness for the property $(\mathcal{G})$ for this sequence.

Final step

To obtain a model with a balanced algebra with the property $(\mathcal{G})$ we extend our algebras ω_1 times using finitely supported iteration of described forcings.

Final step

To obtain a model with a balanced algebra with the property $(\mathcal{G})$ we extend our algebras ω_1 times using finitely supported iteration of described forcings.

In this model we have

$$\mathfrak{p} = \mathfrak{s} = \mathfrak{cov}(\mathcal{M}) = \omega_1$$

Thank you for your attention!