Enriched Hausdorff and Vietoris functors

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[1]based on joint work with Carla Reis, Renato Neves and Pedro Nora.

Definition. For a functor $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$, one defines **coalgebra**

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Theorem. The forgetful functor $\mathrm{CoAlg}(\mathsf{F}) \longrightarrow \mathbf{C}$ creates all colimits and those limits which are preserved by F.

Theorem. The final coalgebra for $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ is a fix-point of F.

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Example. The power-set functor $\mathsf{P}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ does not admit a final coalgebra.

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Example. The finite power-set functor $\mathsf{P}_{\mathrm{fin}}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ admits a final coalgebra (for instance, because $\mathsf{P}_{\mathrm{fin}}$ is finitary).

BARR, MICHAEL (1993). "Terminal coalgebras in well-founded set theory". In: *Theoretical Computer Science* **114**.(2), pp. 299–315.

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- liftings of $\mathbf{Set}$-functors to $\mathbf{Met}$ (or, more general, to $\mathcal{V}\text{-}\mathbf{Cat}$)?

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\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\overline{\mathsf{F}}"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[d] \\
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- (in particular) the Hausdorff functor?

$$\mathsf{H}\colon \mathbf{Met} \longrightarrow \mathbf{Met}$$

Here: $\mathsf{H}a(A,B) = \sup_{y\in B} \inf_{x\in A} a(x,y)$, for a metric space (X,a) and $A,B \subseteq X$.

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- (in particular) the Hausdorff functor?

$$\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

Here: $\mathsf{H}a(A,B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x,y)$, for a $\mathcal{V}$-category (X,a) and $A,B \subseteq X$.

Theorem. Consider the following commutative diagram of functors.

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\overline{\mathsf{F}}"] \arrow[d, "\mathsf{U}"'] & \mathbf{X} \arrow[d, "\mathsf{U}"] \\ \mathbf{A} \arrow[r, "\mathsf{F}"'] & \mathbf{A} \end{tikzcd}$$

1. If $\overline{\mathsf{F}}$ has a fix-point, then so has F. Hence, if F does not have a fix-point, then neither does $\overline{\mathsf{F}}$.

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1. If $\overline{\mathsf{F}}$ has a fix-point, then so has F. Hence, if F does not have a fix-point, then neither does $\overline{\mathsf{F}}$.
2. If $\mathsf{U}\colon \mathbf{X} \longrightarrow \mathbf{A}$ is topological, then so is $\mathsf{U}\colon \mathrm{CoAlg}(\overline{\mathsf{F}}) \longrightarrow \mathrm{CoAlg}(\mathsf{F})$.

   In particular, the category $\mathrm{CoAlg}(\overline{\mathsf{F}})$ has limits of shape I if and only if $\mathrm{CoAlg}(\mathsf{F})$ has limits of shape I.

Theorem. Let X be a partially ordered set. Then there is no embedding $\varphi\colon \mathsf{Up}(X) \longrightarrow X$.

Dilworth, Robert P. and Gleason, Andrew M. (1962). "A generalized Cantor theorem". In: *Proceedings of the American Mathematical Society* **13**.(5), pp. 704–705.

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Corollary. The up-set functor $\mathsf{Up}\colon \mathbf{Ord} \longrightarrow \mathbf{Ord}$ does not admit a final coalgebra.

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3. We consider the $\mathcal{V}$-category $\mathsf{H}X = \{A \subseteq X \mid A \text{ is increasing}\}$, equipped with

$$\mathsf{H}a(A,B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x,y),$$

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for all $A, B \in \mathsf{H}X$.
4. $\mathsf{H}f\colon \mathsf{H}(X,a) \longrightarrow \mathsf{H}(Y,b)$ sends an increasing subset $A \subseteq X$ to $\uparrow^b f(A)$.

We obtain the functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$.

Akhvlediani, Andrei, Clementino, Maria Manuel, and Tholen, Walter (2010). "On the categorical meaning of Hausdorff and Gromov distances, I". In: *Topology and its Applications* **157**.(8), pp. 1275–1295.

Stubbe, Isar (2010). ""Hausdorff distance" via conical cocompletion". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **51**.(1), pp. 51–76.

> ***Theorem.*** Let $\mathcal{V}$ be a non-trivial quantale and (X, a) be a $\mathcal{V}$-category. There is no embedding of type $\mathsf{H}(X, a) \longrightarrow (X, a)$.

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Corollary. Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}\text{-}\mathbf{Cat}$.

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> *"A cardinal principle of modern mathematical research may be stated as a maxim: One must always topologize."*

Stone, Marshall Harvey (1938). "The representation of Boolean algebras". In: *Bulletin of the American Mathematical Society* **44**.(12), pp. 807–816.

Theorem. If the $\mathbf{C}$ has and $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ preserves the limit L of the diagram

$$1 \longleftarrow \mathsf{F}1 \longleftarrow \mathsf{FF}1 \longleftarrow \ldots,$$

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Definition. $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ is a **covarietor** if $\mathrm{CoAlg}(\mathsf{F}) \longrightarrow \mathbf{C}$ is left adjoint.

Theorem. If $\mathbf{C}$ is cocomplete with finite limits and $\mathbf{C}$ has and $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ preserves limits of countable chains, then $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ is a covarietor.

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Proof. F is a covarietor iff $F(-) \times X$ has a terminal coalgebra, for all X. □

Theorem. Let F be a covarietor over a complete category. If $\mathrm{CoAlg}(\mathsf{F})$ has equalisers then $\mathrm{CoAlg}(\mathsf{F})$ is complete.

Linton, F. E. J. (1969). "Coequalizers in categories of algebras". In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. Eckmann. Vol. 80. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 75–90.

Theorem. Let $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ be an endofunctor on a cocomplete category $\mathbf{C}$ and let I be a small category. If $\mathbf{C}$ is $(E, \mathcal{M})$-structured for cones for I, $\mathcal{M}$-wellpowered and F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$, then $\mathrm{CoAlg}(\mathsf{F})$ has limits of shape I.

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Example. If $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ preserves monocones of a certain type, then the category $\mathrm{CoAlg}(F)$ has limits of the same type.

Example. If $F\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ preserves either small monocones or small initial monocones of a certain type, then the category $\mathrm{CoAlg}(F)$ has limits of the same type.

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Corollary. Let $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ be an endofunctor on a cocomplete category $\mathbf{C}$. If $\mathbf{C}$ is regularly wellpowered, has an (Epi,RegMono)-factorisation structure and $\mathsf{F}\colon \mathbf{C} \longrightarrow \mathbf{C}$ preserves regular monos, then $\mathrm{CoAlg}(\mathsf{F})$ has equalisers.

For a compact Hausdorff space X, the **classic Vietoris space** $\mathsf{V}X$ consists of the set of all closed subsets of X

$$\mathsf{V}X = \{K \subseteq X \mid K \text{ is closed } (= \text{compact})\}$$

equipped with the "hit-and-miss topology" generated by the subbasis of sets of the form (where $U \subseteq X$ is open)

$$U^{\Diamond} = \{A \in \mathsf{V}X \mid A \cap U \neq \varnothing\} \qquad (\text{“}A \text{ hits } U\text{”}),$$
$$U^{\Box} = \{A \in \mathsf{V}X \mid A \cap U^{\complement} = \varnothing\} \qquad (\text{“}A \text{ misses } U^{\complement}\text{”}).$$

We obtain $\mathsf{V} \colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$.

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> *Remark.* This definition can be generalised to arbitrary topological spaces … but does not always defines a functor!!

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Remark. We consider here the following two variants on $\mathbf{Top}$:

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- **lower Vietoris**: closed subsets, but only "hit topology".
- **compact Vietoris**: compact subsets, "hit-and-miss topology".

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- The compact Vietoris functor preserves initial cofiltered monocones of Hausdorff spaces. Therefore $\mathrm{CoAlg}(\mathsf{V})$ has cofiltered limits of Hausdorff spaces.

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- The compact Vietoris functor preserves initial cofiltered monocones of Hausdorff spaces. Therefore $\mathrm{CoAlg}(\mathsf{V})$ has cofiltered limits of Hausdorff spaces.
- $\mathsf{V}\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ preserves regular monomorphisms.
- The category $\mathrm{CoAlg}(\mathsf{V})$ has equalisers.

Theorem. The compact Vietoris functor $\mathsf{V}\colon \mathbf{Haus} \longrightarrow \mathbf{Haus}$ preserves cofiltered limits and closed embeddings (= regular monos). Hence, $\mathrm{CoAlg}(\mathsf{V})$ is complete.

Zenor, Phillip (1970). "On the completeness of the space of compact subsets". In: *Proceedings of the American Mathematical Society* **26**.(1), pp. 190–192.

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Theorem. The classic Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$ preserves cofiltered limits and embeddings. Hence, $\mathrm{CoAlg}(\mathsf{V})$ is complete.

Theorem. Let $D\colon I \longrightarrow \mathbf{CompHaus}$ be a cofiltered diagram. Then a cone $(p_i\colon L \longrightarrow D(i))_{i\in I}$ for D is a limit cone if and only if

1. $(p_i\colon L \longrightarrow D(i))_{i\in I}$ is mono and,
2. for every $i \in I$:
$$\bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i.$$
   That is, "the image of each p_i is as large as possible".

Bourbaki, Nicolas (1942). *Éléments de mathématique. 3. Pt. 1: Les structures fondamentales de l'analyse. Livre 3: Topologie générale*. Paris: Hermann & Cie.

Remark. The lower Vietoris functor $\mathsf{V}\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ restricts to an endofunctor on the category $\mathbf{StablyComp}$ of stably compact spaces and spectral maps.

$$\begin{tikzcd} \mathbf{CompHaus} \arrow[r, bend left, "\text{discrete}"] \arrow[r, phantom, "\perp"] & \mathbf{PosComp} \sim \mathbf{StablyComp} \arrow[l, bend left, "\text{forget}"] \end{tikzcd}$$

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Theorem. The lower Vietoris functors $\mathsf{V}\colon \mathbf{StablyComp} \longrightarrow \mathbf{StablyComp}$ preserve cofiltered limits and embeddings. Hence, $\mathrm{CoAlg}(\mathsf{V})$ is complete.

Remark. The lower Vietoris functor $\mathsf{V}\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ restricts to an endofunctor on the category $\mathbf{StablyComp}$ of stably compact spaces and spectral maps.

$$\begin{tikzcd} \mathbf{CompHaus} \arrow[r, bend left, "\text{discrete}"] \arrow[r, phantom, "\perp"] & \mathbf{PosComp} \sim \mathbf{StablyComp} \arrow[l, bend left, "\text{forget"] \end{tikzcd}$$

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Now comes a little surprise (at least to us):

Corollary. The lower Vietoris functor $\mathsf{V}\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ admits a terminal coalgebra.

Proof. Use that $\mathbf{StablyComp} \longrightarrow \mathbf{Top}$ is closed under limits and

$$1 \longleftarrow \mathsf{V}1 \longleftarrow \mathsf{VV}1 \longleftarrow \dots$$

lives in $\mathbf{StablyComp}$. $\square$

We assume that $\mathcal{V}$ is a completely distributive quantale, then

$$\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}, \quad \mathfrak{v} \longmapsto \bigwedge_{A\in\mathfrak{v}} \bigvee A$$

is the structure of an $\mathbb{U}$-algebra on $\mathcal{V}$ (the Lawson topology).

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- We obtain a lax extension of the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ to $\mathcal{V}$-$\mathbf{Rel}$ that induces a monad on $\mathcal{V}$-$\mathbf{Cat}$.

$$\mathsf{U}a(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A,B} \bigvee_{x,y} a(x,y), \qquad (X,a) \longmapsto (\mathsf{U}X, \mathsf{U}a).$$

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- Its algebras are $\mathcal{V}$-categories equipped with a compatible compact Hausdorff topology, called **$\mathcal{V}$-categorical compact Hausdorff spaces**.
- We denote the corresponding Eilenberg–Moore category by $\mathcal{V}$-$\mathbf{CatCH}$.

Theorem. For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \longrightarrow (X, a)$ is a $\mathcal{V}$-functor.

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Theorem. For an ordered set $(X, \leq)$ and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon (\mathsf{U}X, \mathsf{U}{\leq}) \longrightarrow (X, \leq)$ is monotone.

(ii) $G_{\leq} \subseteq X \times X$ is closed; that is, $\chi_{\leq}\colon X \times X \longrightarrow 2$ is continuous.

Tholen, Walter (2009). "Ordered topological structures". In: *Topology and its Applications* **156**.(12), pp. 2148–2157.

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Theorem. For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \longrightarrow (X, a)$ is a $\mathcal{V}$-functor.

(ii) $a\colon (X, \alpha) \times (X, \alpha) \longrightarrow (\mathcal{V}, \xi_{\leq})$ is continuous.

Definition. For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$\mathsf{H}X = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

with the restriction of the Hausdorff structure to $\mathsf{H}X$ and the hit-and-miss topology (Vietoris topology).

Remark. That is, the topology generated by the sets

$$V^{\Diamond} = \{A \in \mathsf{H}X \mid A \cap V \neq \varnothing\} \qquad (V \text{ open, co-increasing})$$

and

$$W^{\Box} = \{A \in \mathsf{H}X \mid A \subseteq W\} \qquad (W \text{ open, co-decreasing}).$$

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Theorem. For every $\mathcal{V}$-categorical compact Hausdorff space X, $\mathsf{H}X$ is a $\mathcal{V}$-categorical compact Hausdorff space.

Compare with: For a compact metric space, the Hausdorff metric induces the Vietoris topology.

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Theorem. The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}$ preserves cofiltered limits. Therefore the forgetful functor $\mathrm{CoAlg}(\mathsf{H}) \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}$ is comonadic. Moreover, $\mathrm{CoAlg}(\mathsf{H})$ has equalisers and is therefore complete.

The order case:

$$\begin{tikzcd}
\mathbf{Ord}^{\mathbb{U}} \arrow[r, "K"] \arrow[dr] & \mathbf{Top} \arrow[d, "(-)_0"] \\
& \mathbf{Ord}
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Remark. $\mathcal{U}$-**distributor** $\varphi\colon X \longrightarrow\!\!\!\!\!\!\circ\;\; Y \;=\; \varphi\colon \mathsf{U}X \times Y \longrightarrow \mathcal{V}$ so that ...

- $\varphi\colon 1 \longrightarrow\!\!\!\!\!\!\circ\;\; X \quad = \quad \mathcal{U}$-functor $\varphi\colon X \longrightarrow \mathcal{V}$.
- $\psi\colon X \longrightarrow\!\!\!\!\!\!\circ\;\; 1 \quad = \quad \mathcal{U}$-functor $\psi\colon (UX)^{\mathrm{op}} \longrightarrow \mathcal{V}$.

Definition. X is **Cauchy complete** if every adjunction $\varphi \dashv \psi$ is induced by some $x \in X$.

Theorem. Under some conditions on $\mathcal{V}$.

1. Every $\mathcal{U}$-category in the image of

$$K\colon (\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{U}} \longrightarrow \mathcal{U}\text{-}\mathbf{Cat}$$

   is Cauchy complete.

2. $(-)_0\colon \mathcal{U}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ sends Cauchy complete $\mathcal{U}$-categories to Cauchy complete $\mathcal{V}$-categories.

3. For every (X, a_0, α) in $(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{U}}$, the $\mathcal{V}$-category (X, a_0) is Cauchy complete.

$$\begin{tikzcd}
(\mathcal{V}\text{-}\mathbf{Cat})^{\mathbb{U}} \arrow[r, "K"] \arrow[dr] & \mathcal{U}\text{-}\mathbf{Cat} \arrow[d, "(-)_0"] \\
& \mathcal{V}\text{-}\mathbf{Cat}
\end{tikzcd}$$

Under some conditions on $\mathcal{V}$.

Definition. A $\mathcal{V}$-distributor $\varphi_0 \colon 1 \longrightarrow\!\!\!\!\!\!\circ\;\;\; X$ is called **codirected** whenever the $\mathcal{V}$-functor

$$[\varphi_0, -] \colon \mathcal{V}\text{-}\mathbf{Dist}(1, X) \longrightarrow \mathcal{V}$$

preserves finite suprema and tensors. A $\mathcal{V}$-category X is called **codirected complete** whenever X has all "codirected" weighted limits.

Theorem. The inclusion $\mathcal{V}$-functor

$$\mathcal{U}\text{-}\mathbf{Dist}(1, X) \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}(1, X_0)$$

has a left adjoint $\overline{(-)} \colon \mathcal{V}\text{-}\mathbf{Dist}(1, X_0) \longrightarrow \mathcal{U}\text{-}\mathbf{Dist}(1, X)$ and $\mathcal{U}\text{-}\mathbf{Dist}(1, X)$ is closed in $\mathcal{V}\text{-}\mathbf{Dist}(1, X_0)$ under finite suprema and tensors.

Corollary. For every codirected $\mathcal{V}$-distributor $\varphi \colon 1 \longrightarrow\!\!\!\!\!\!\circ\;\;\; X_0$, the $\mathcal{U}$-distributor $\overline{\varphi} \colon 1 \longrightarrow\!\!\!\!\!\!\circ\;\;\; X$ is left adjoint in $\mathcal{U}\text{-}\mathbf{Dist}$.

Corollary. For every $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a_0, \alpha)$, the $\mathcal{V}$-categories (X, a_0) and $\mathsf{H}X$ are codirected complete.