On limit sets and equicontinity in the hyperspace of continua in dimension one

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Hyperspaces

A **hyperspace** of a compact metric space (X, d) is a specified family of nonempty compact subsets of X. The hyperspaces of our interest are:

- $\boldsymbol{2^X}$, the hyperspace of nonempty compact subsets of X and
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We endow 2^X with **Hausdorff metric** d_H:

$$\left(\forall A, B \in 2^X\right)\ \mathrm{d}_H(A, B) = \inf\{\epsilon \geq 0\colon A \subset \mathrm{N}(B, \epsilon) \text{ and } B \subset \mathrm{N}(A, \epsilon)\}.$$

The induced map

Let X be a compact metric space and $f\colon X \to X$ be a continuous map

1. f naturally extends to $\bar{f}\colon 2^X \to 2^X$ via $\bar{f}(A) = f[A]$
   - $\bar{f}$ is a continuous map on 2^X

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   - $\bar{f}$ is a continuous map on 2^X
   - $\mathcal{C}(X)$ is $\bar{f}$-invariant
2. we denote $\tilde{f} = \bar{f}|_{\mathcal{C}(X)}$ and call it the **induced map**.

Question

Let (X, f) be a dynamical system. What are the properties of the induced system $(\mathcal{C}(X), \tilde{f})$ on the hyperspace of continua in X? How are they related to the base map?

Topological graph

1. A **topological graph** is a continuum G such that there is a one dimensional simplical complex K whose geometric carrier $|K|$ is homeomorphic to G.
2. A **subgraph** is any nondegenerate subcontinuum of a graph.

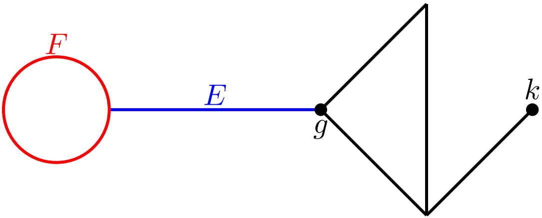


Figure: E-edge, g-branching point, k-endpoint, F-loop

Topological tree

3. Topological graph not containing any homeomorphic copy of a circle is called a **(topological) tree**.

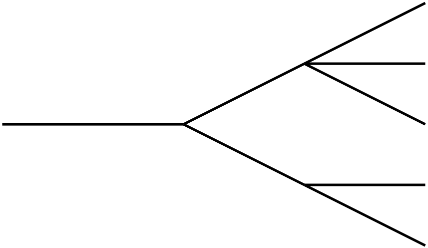


Figure: A tree

A few more definitions...

Let (X, f) be a dynamical system and $x_0 \in X$.

- The **$\boldsymbol{\omega}$-limit set of $\boldsymbol{x_0}$**, denoted by $\omega_f(x_0)$, is the limit set of its trajectory.
- We say that x_0 is **recurrent** if $x_0 \in \omega_f(x_0)$. The set of all recurrent points of (X, f) is denoted by $\mathrm{Rec}(f)$.
- The **Birkhoff center of $\boldsymbol{(X, f)}$** is the set $\mathcal{C}(f) = \overline{\mathrm{Rec}(f)}$.

A few more definitions...

- x_0 is said to be **equicontinuous** if

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall y \in B(x_0, \delta))\ (\forall n \geq 0)\ \mathrm{d}(f^n(y), f^n(x_0)) < \epsilon.$$

- (X, f) is said to be **equicontinuous** if the set $\mathcal{E}$ of all the equicontinuous points equals X and **almost equicontinuous** if $\mathcal{E}$ is dense and G_δ in X.

Earlier results

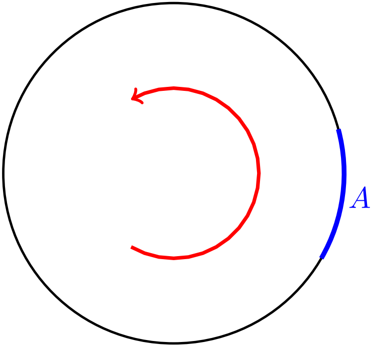
Theorem (Matviichuk)

Let $f : T \to T$ be a tree map. Then each $A \in \mathcal{C}(T)$ is asymptotically periodic under $\tilde{f}$ or asymptotically degenerate (or both).

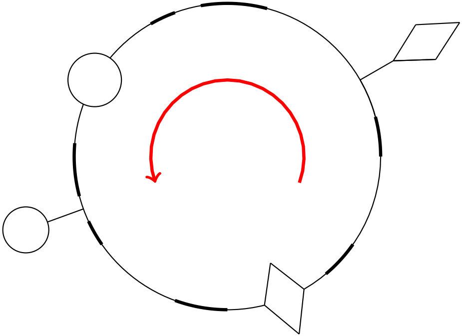
Corollary

Let $f : T \to T$ be a tree map. Then $\mathrm{h}_{\mathrm{top}}(f) = \mathrm{h}_{\mathrm{top}}(\tilde{f})$.

Generalizing to graph maps



(a) Irrational rotation



(b) Almost 1-1 extension of irr. rot.

Generalizing to graph maps

- existence of circles makes proofs more technical
- too many cases depending on all the possible geometrical structures of graphs
- solution: considering relations between subgraphs and the α and ω-limit sets of the points forming them

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Theorem (J., Oprocha)

Let G be a topological graph and let $(\mathcal{C}(G), \tilde{f})$ be its induced system on the hyperspace of continua. Then every subgraph of G is either asymptotically periodic, wandering or almost all of its iterates are contained in a subsystem which is an almost 1-1 extension of irrational rotation.

Periodic subcontinua

Theorem (Fedorenko)

If a nondegenerate interval $[a, b] \in \mathcal{C}(I)$ contains a preimage of a periodic point of period p then $[a, b]$ is asymptotically periodic of period which is a divisor of $2p$.

Question

Do the topological graphs share some analogous property? Or at least topological trees?

Motivation

1. Describing Birkhoff center of the induced system $(\mathcal{C}(T), \tilde{f})$.
   - What does the closure of the set $\mathrm{Per}(\tilde{f})$ equal?
   - Can we have overlaping periodic subontinua of arbitrary periods?
2. Proving almost equicontinuity of the induced system.

Theorem

Let T be a tree and $f : T \to T$ its continuous selfmap. Let $A \in \mathcal{C}(T)$ be the periodic point of $\tilde{f}$ containing an element z of T such that $f(z) = z$. Then the period of A is a divisor of $\operatorname{lcm}\{1, 2, ..., |\operatorname{End}(T)|\}$.

Corollary

Let $f : T \to T$ be a continuous selfmap of a topological tree. Denote $m = \operatorname{lcm}\{1, 2, ..., |\operatorname{End}(T)|\}$ and let P_1 and P_2 be two intersecting periodic points of $(\mathcal{C}(T), \tilde{f})$ of periods p_1 and p_2, respectively. If $p_1 > mp_2$ then $P_1 \subset P_2$.

Consequences

Theorem (Robatian)

Let $f\colon I \to I$ be an interval map. Then $(\mathcal{C}(I), \tilde{f})$ possesses an equicontinuous point.

Theorem

Let $f\colon T \to T$ be a continuous selfmap of a tree. Then $(\mathcal{C}(T), \tilde{f})$ is almost equicontinuous.

Consequences

Theorem (Matviichuk)

Let $f\colon I \to I$ be an interval map. Then

$$\overline{\mathrm{Per}(\tilde{f})} = \left\{\{x\}\colon x \in \overline{\mathrm{Per}(f)}\right\} \cup \mathrm{Per}(\tilde{f}) \cup T,$$

where T is a set of pairwise disjoint nondegenerate asymptotically degenerate intervals.

Theorem

Let f be a continuous selfmap of a topological tree T. Then

$$\mathrm{C}\left(\tilde{f}\right) = \{\{x\}\colon x \in \mathrm{C}(f)\} \cup \mathrm{Per}\left(\tilde{f}\right) \cup \mathcal{S},$$

where $\mathcal{S}$ consists of wandering subtrees, attracted by solenoidal ω-limit sets.

Thank you!