Spaces with an *M*-diagonal

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Contents

Definition

A space Z is (strongly) dominated by a space Y if there exists a family $\mathcal{D} = \{D_K : K \text{ is a compact subset of } Y\}$ such that $\bigcup \mathcal{D} = Z$, every D_K is compact and $K \subset L$ implies $D_K \subset D_L$ (and for every compact $F \subset Z$ there is a compact $K \subset Y$ such that $F \subset D_K$). We will say that a space X has an M-diagonal (or a strong M-diagonal) if $(X \times X) \setminus \Delta$ is (strongly) dominated by the space M; here $\Delta = \{(x, x) : x \in X\}$ is the diagonal of the space X.

Christensen [CH 74] proved that a second countable space is strongly M-dominatedby a Polish space if and only if it is completely metrizable.

In [CO, 87] B. Cascales and J. Orihuela proved (using a different terminology) that a compact space X is metrizable whenever it has a $\mathbb{P}$-diagonal.

Later, in [Tk 05], V. Tkachuk systematically studied the spaces that are (strongly) dominated by the irrationals and called them $\mathcal{P}$-dominated spaces.

B. Cascales, J. Orihuela and V. Tkachuk [COT 2011] proved that if X is a compact space of countable tightness with an M-diagonal for some second countable space M, then X is metrizable.

A. Dow and K. P. Hart [DH 2016] showed that any compact space with a $\mathbb{P}$-diagonal is metrizable. Here $\mathbb{P}$ is the space of the irrational numbers.

in [Fe 2018] Z. Feng proved that every compact space with a

Previous results

D. Basile and D. Udayan. wrote in [BD 2017]: “In recent years much attention has been enjoyed by topological spaces which are dominated by second countable spaces". To this regard, we refer to the work of P. Gartside and A. Mamatelashvilli [GM 2016], C. Islas and D. Jardón [IJ 2019], J.Kakol, M. López Pellicer and O. Okunev [KLO 2014], besides the mentioned paper [BD 2017].

With regard to [COT, Problem 4.3 2011] specifically, we can mention the paper [GMo 2017] where P. Gartside and J. Morgan proved that a compact space is metrizable whenever the complement of its diagonal in its square has caliber $(\omega_1, \omega, \omega)$. Also, in [DG], D. Guerrero and A. Dow proved that assuming CH a compact space X is metrizable whenever it has a $\mathbb{P}$-diagonal, and later, the same conclusion was obtained in ZFC by A. Dow and K.P. Hart in [DH 2016]. In [GT 2016], D. Guerrero and V. Tkachuk then generalized the result in [DG 2015] and proved that assuming CH a compact space X is

Previous results

The evident importance of the research conducted in [COT] is that it provides with metrizability conditions for several sorts of topological spaces such as compact spaces and function spaces. For instance, in [COT] the authors generalized the aforementioned result published in [CO] by proving that a compact space X is metrizable whenever it has a strong M-diagonal for some separable metric space M. About that result the authors commented: “One of the niceties of the concept of domination by a second countable space is a possibility to obtain new metrization theorems for compact spaces. We already saw that if X compact and $(X \times X) \setminus \Delta$ is strongly dominated by a second countable space then X is metrizable. The most interesting question here is whether we can omit the word ‘strongly’ in the above statement"(see [COT, Problem 4.3]). B. Cascales, J. Orihuela and V. Tkachuk themselves obtained some partial answer to their question when they showed in their paper that every compact space with

Main Lemma

Suppose X is a countably compact non-compact space with an unbounded partial order (that does not necessarily generates the topology of X) Additionally, asume that a set K contained in X is compact if and only if K is closed and bounded in X; and every non-cofinal subset of X is contained in a compact subset of X. Given these conditions, for any $C \subset X$ the set $\overline{C}$ is cofinal in X provided C itself is cofinal in X; also, if X is dominated by a second countable space M and $\{F_K : K \in \mathcal{K}(M)\}$ is a family that witnesses such domination, then we can find families $\mathcal{C} = \{C_K : K \in \mathcal{K}(M)\}$ and $\mathcal{N} = \{N_n : n \in \omega\}$ such that:

Main Lemma

1. The family $\mathcal{C} = \{C_K : K \in \mathcal{K}(M)\}$ is a cover of X for which $K \subset L$ implies $C_K \subset C_L$;
2. if $C \in \mathcal{C}$ and $A \subset C$ is countable, then $\overline{A}$ is a compact subset of C (i.e. the elements of $\mathcal{C}$ are ω-bounded); in particular, each $C \in \mathcal{C}$ is countably compact;
3. for every $K \in \mathcal{K}(M)$ the set $F_K \in C_K$;
4. the family $\mathcal{N}$ is a network with respect to $\mathcal{C}$;
5. there exists $K \in \mathcal{K}(M)$ such that C_K is cofinal in X.

Theorem.

Suppose that X is a compact space with an M-diagonal for some second countable space M. Then X is metrizable

Corollary.

If M is a separable metric space then every Tychonoff space X with an M-diagonal is cosmic.

Corollary.

Let M be a metric space and X be a compact space with an M-diagonal. If X has small diagonal, then $w(X) \leq w(M)$.

Theorem.

If M is a metric space and X be a compact space with a strong M-diagonal, then $t(X) \leq w(M)$.

Open problems

1. Suppose M is a metric space (not necessarily separable) and K is a compact space with an M-diagonal. Is it true that $w(K) \leq w(M)$?
2. Suppose M is a metric space (not necessarily separable) and X is a Tychonoff space with M-diagonal. Is it true that $nw(X) \leq w(M)$?

For any compact space X, if $(X \times X) \setminus \Delta$ is Lindelöf, then the space X has a G_δ-diagonal so it must be metrizable. This makes it natural to ask the following questions.

1. Suppose κ is an uncountable cardinal, the space Y is the union of κ-many compact spaces and X is a compact space such that $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?
2. Suppose κ is an uncountable cardinal, the Lindelöf degree of the space Y is κ and X is a compact space such that $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?

Open problems

1. Suppose κ is an uncountable cardinal, the hereditary Lindelöf degree of the space Y is κ and X is a compact space such that $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?
2. Suppose κ is an uncountable cardinal, the network weight of the space Y is κ and X is a compact space such that $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?

Open problems

Recall that if X is a compact space of countable tightness and $(X \times X) \setminus \Delta$ is dominated by a metric space of weight κ, then $w(X) \leq \kappa$. This motivates the next group of questions.

1. Suppose κ is an uncountable cardinal, the space Y is the union of κ-many compact spaces and X is a compact space such that $t(X) \leq \kappa$ and $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?
2. Suppose κ is an uncountable cardinal, the Lindelöf degree of the space Y is κ and X is a compact space such that $t(X) \leq \kappa$ and $(X \times X)\Delta$ is dominated by Y. Must $w(X) \leq \kappa$?

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