# Topologies on real-enriched categories

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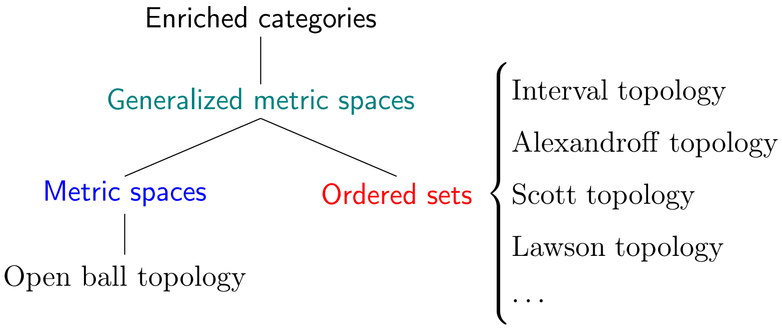
The story begins with the investigation of topological structures of generalized metric spaces.




A generalized metric is a quasi-pseudo-metric with distance allowed to be infinite.

Differences between metric spaces and ordered sets

- symmetry
- enrichment (viewed as enriched categories)

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Topologies of a generalized metric space

- open ball topology, which makes use of the Scott topology of the enrichment $([0, \infty], \geq)$
- Scott & Lawson type topologies, which are motivated by the Scott and the Lawson topology of ordered sets

A generalized metric on a set $X$ is a function $d\colon X \times X \longrightarrow [0, \infty]$ such that for all $x, y, z \in X$,

(i) $d(x, x) = 0$;
(ii) $d(y, z) + d(x, y) \geq d(x, z)$.

The open ball topology of $(X, d)$ is generated as a basis by $\{B(x, r) \mid x \in X, r > 0\}$, where $B(x, r)$ is the open ball with center $x$ and radius $r$: $\{y \in X \mid d(x, y) < r\}$.

Every topology with a countable basis is the open ball topology of a generalized metric.

W.A. Wilson, On quasi-metrizable spaces, American Journal of Mathematics 53 (1931) 675–684.

Because of the asymmetry, in 1963 Kelly considered two topologies on $X$, which are generated respectively by the 'forward open balls' and the 'backward open balls'. Kelly suggested that "the natural topological structure associated with a quasi-pseudo-metric on a set $X$ is that of the set $X$ with two topologies."

The backward open ball with center $x$ and radius $r$ is the set

$$\{y \in X \mid d(y, x) < r\}.$$

The join of the two topologies of $(X, d)$ is the open ball topology of its symmetrization.

J.C. Kelly, Bitopological spaces, Proceedings of the London Mathematical Society 13 (1963) 71-89.

1973, Lawvere emphasized that generalized metric spaces are categories enriched over the quantale $([0,\infty],\geq,+,0)$. Parallel concepts and results abound in the theories of ordered sets, generalized metric spaces, and enriched categories.

The analogy between ordered sets and generalized metric spaces leads people to extend the construction of Scott topology of ordered sets to generalized metric spaces, and more generally, to quantale-enriched categories.

The Scott topology of an ordered set $(P, \leq)$ encodes the order relation and makes every directed subset (viewed as a self-indexed net) converge to its join. It is determined by convergence of directed subsets.

Precisely, a subset $U$ of $(P, \leq)$ is Scott open if

(i) for each directed subset $D$ of $P$, $\sup D \in U$ implies that $D$ is eventually in $U$;

(ii) $U$ is an upper set; or equivalently, the characteristic function $\chi_U \colon P \longrightarrow \{0, 1\}$ is a functor, when $(P, \leq)$ is viewed as a category.

D.S. Scott, Continuous lattices, Lecture Notes in Mathematics, Volume 274, Springer, 1972, pp. 97-136.

# Some existing works

- g-Scott (generalized Scott) topology for quantale-enriched categories

  M.M. Bonsangue, F. van Breugel, J.J.M.M. Rutten, Generalized metric space: completion, topology, and powerdomains via the Yoneda embedding, Theoretical Computer Science 193 (1998) 1-51.

  R.C. Flagg, P. Sünderhauf, K.R. Wagner, A logical approach to quantitative domain theory, Topology Atlas Preprint No. 23, 1996.

- d-Scott topology for generalized metric spaces (d for the metric $d$)

  J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory, Cambridge University Press, Cambridge, 2013.

# What we'll do

We consider topological structures on V-categories with V being the quantale of $[0, 1]$ together with a continuous t-norm &.

The idea is to construct V-topological structures first, then topologies by means of these V-topological structures.

V-topological spaces, extensions of approach spaces to the quantale-valued setting, are topological spaces in the enriched context.

| ordered sets | topological spaces |
|---|---|
| V-categories | V-topological spaces |

R. Lowen, Approach spaces: a common supercategory of TOP and MET, Mathematische Nachrichten 141 (1989) 183-226.

H. Lai, W. Tholen, Quantale-valued topological spaces via closure and convergence, Topology and its Applications 230 (2017) 599-620.

In particular, the c-Scott topology of a V-category will be defined to be the topological modification of its Scott V-topology:

$$
\begin{tikzcd}
\mathsf{V}\text{-}\mathsf{Cat}^{\uparrow} \arrow[r, "\Sigma"] \arrow[dr, "\text{c-Scott}"'] & \mathsf{V}\text{-}\mathsf{Top} \arrow[d, "\iota"] \\
 & \mathsf{Top}
\end{tikzcd}
$$

# Continuous t-norm

A continuous t-norm on $[0,1]$ is a continuous function

$$\& \colon [0,1] \times [0,1] \longrightarrow [0,1],$$

called multiplication, such that $([0,1], \&, 1)$ is a lattice-ordered monoid; that means,

(i) $([0,1], \&, 1)$ is a monoid;
(ii) $\&$ is continuous and non-decreasing on each variable.

✎ It is known that every continuous t-norm is commutative.

For a continuous t-norm & on $[0,1]$, define a function

$$\to : [0,1] \times [0,1] \longrightarrow [0,1]$$

and call it the implication of &, as follows:

$$x \to z = \sup\{y \mid x \,\&\, y \le z\}.$$

By continuity of & we have the adjoint property

$$x \,\&\, y \le z \iff y \le x \to z.$$

Because of this property, continuous t-norms are often used to model the logic connective conjunction in many-valued logics.

## Example 1 (Basic continuous t-norms and their implications)

- The Gödel t-norm:

$$x \,\&\, y = \min\{x, y\}, \quad x \to z = \begin{cases} 1 & x \le z, \\ z & x > z. \end{cases}$$

  The quantale $([0,1], \min, 1)$ is isomorphic to $([0,\infty], \max, 0)$.

- The product t-norm:

$$x \,\&\, y = x \cdot y, \quad x \to z = \begin{cases} 1 & x \le z, \\ z/x & x > z. \end{cases}$$

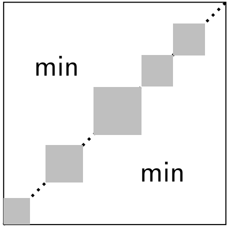
  The quantale $([0,1], \cdot, 1)$ is isomorphic to Lawvere's quantale $([0,\infty], +, 0)$.

- The Łukasiewicz t-norm:

$$x \,\&\, y = \max\{x + y - 1, 0\}, \quad x \to z = \min\{1 - x + z, 1\}.$$

## Theorem (Mostert and Shields, 1957)

Every continuous t-norm on $[0, 1]$ is an ordinal sum of the product t-norm and the Łukasiewicz t-norm.




For each continuous t-norm &, there exists a family of pairwise disjoint (except possibly at endpoints) subintervals of $[0, 1]$ such that the restriction of & on each subinterval is a continuous t-norm that is isomorphic either to the product or to the Łukasiewicz t-norm, outside those squares & is equal to the Gödel t-norm.

Let & be a continuous t-norm on the interval $[0, 1]$. The triple

$$\mathsf{V} = ([0, 1], \&, 1)$$

is then a commutative and unital quantale, hence a complete, skeletal, symmetric, and monoidal closed category.

Categories enriched over such a quantale are called real-enriched categories.

# The category of real-enriched categories

In this talk & always denotes a continuous t-norm on the interval $[0,1]$ and V denotes the quantale $([0,1],\&,1)$.

A real-enriched category (a V-category, to be precise) is a pair $(X,\alpha)$, where $X$ is a set and $\alpha\colon X\times X\longrightarrow[0,1]$ is a function such that for all $x,y,z\in X$,

(i) $\alpha(x,x)=1$;
(ii) $\alpha(y,z)\,\&\,\alpha(x,y)\leq\alpha(x,z)$.

To ease notations, we write $X(x,y)$ for $\alpha(x,y)$.

## Example 2

- For all $x, y \in [0,1]$, let
$$\alpha_L(x,y) = x \to y.$$
Then $([0,1], \alpha_L)$ is a real-enriched category.
- The opposite category of $([0,1], \alpha_L)$, written $([0,1], \alpha_R)$, is given by
$$\alpha_R(x,y) = y \to x.$$

The V-categories $([0,1], \alpha_L)$ and $([0,1], \alpha_R)$ are denoted by $\mathsf{V}$ and $\mathsf{V}^{\mathrm{op}}$, respectively.

A functor $f\colon X \longrightarrow Y$ between real-enriched categories is a map such that for all $x, y \in X$,

$$X(x,y) \leq Y(f(x), f(y)).$$

Write

$$\text{V-Cat}$$

for the category of real-enriched categories and functors.

# The category of V-topological spaces

A V-*topology* (or, a V-*approach structure*) on a set $X$ is a map

$$\delta\colon X \times 2^X \longrightarrow [0,1]$$

such that for all $x \in X$ and $A, B \in 2^X$,

(A1) $\delta(x, \{x\}) = 1$;

(A2) $\delta(x, \varnothing) = 0$;

(A3) $\delta(x, A \cup B) = \delta(x, A) \vee \delta(x, B)$;

(A4) $\delta(x, A) \geq \left(\inf_{b \in B} \delta(b, A)\right) \,\&\, \delta(x, B)$.

The pair $(X, \delta)$ is called a V-topological space. The value $\delta(x, A)$ measures the truth that $x$ is in the closure of $A$.

A *continuous map* $f\colon (X,\delta_X)\longrightarrow(Y,\delta_Y)$ between V-topological spaces is a map such that for all $x\in X$ and $A\subseteq X$,

$$\delta_X(x,A)\le\delta_Y(f(x),f(A)).$$

Write

$$\mathsf{V}\text{-}\mathsf{Top}$$

for the category of V-topological spaces and continuous maps.

R. Lowen, Index Analysis, Approach Theory at Work, Springer, 2015.

H. Lai, W. Tholen, Quantale-valued topological spaces via closure and convergence, Topology and its Applications 230 (2017) 599-620.

### Example 3 (The space $\mathbb{K} = ([0,1], \delta_{\mathbb{K}})$)

The map

$$\delta_{\mathbb{K}} \colon [0,1] \times 2^{[0,1]} \longrightarrow [0,1], \quad \delta_{\mathbb{K}}(x, A) = \begin{cases} \inf A \to x & A \neq \varnothing, \\ 0 & A = \varnothing \end{cases}$$

is a V-topology on $[0,1]$.

The space $\mathbb{K} = ([0,1], \delta_{\mathbb{K}})$ plays a role in the category of V-topological spaces analogous to that of the Sierpiński space in the category of topological spaces.

A closed set of a V-topological space $(X, \delta)$ is a continuous function

$$\lambda \colon (X, \delta) \longrightarrow ([0,1], \delta_{\mathbb{K}}).$$

## Proposition 4 (V-topology via closed sets)

For each V-topological space $(X, \delta)$, the set $\mathcal{C}_\delta$ of closed sets satisfies:

(C1) $\lambda, \mu \in \mathcal{C}_\delta \implies \lambda \vee \mu \in \mathcal{C}_\delta$;

(C2) $\lambda \in \mathcal{C}_\delta \implies p \,\&\, \lambda \in \mathcal{C}_\delta$ for all $p \in [0,1]$;

(C3) $\{\lambda_i\}_{i\in I} \subseteq \mathcal{C}_\delta \implies \inf_{i\in I} \lambda_i \in \mathcal{C}_\delta$;

(C4) $\lambda \in \mathcal{C}_\delta \implies p \to \lambda \in \mathcal{C}_\delta$ for all $p \in [0,1]$.

Conversely, if $\mathcal{C} \subseteq [0,1]^X$ satisfies (C1)-(C4), then there is a unique V-topology $\delta$ on $X$ such that $\mathcal{C}$ is its set of closed sets.

The category V-Top contains both the category of topological spaces and the category of V-categories as coreflective full subcategories.

$$\begin{tikzcd} & \mathsf{V\text{-}Cat} \arrow[d, shift right, "\Gamma"'] \\ \mathsf{Top} \arrow[r, shift left, "\omega"] & \mathsf{V\text{-}Top} \arrow[l, shift left, "\iota"] \arrow[u, shift right, "\Omega"'] \end{tikzcd}$$

$\omega$: a full and faithful functor

$\iota$: topological modification

$\Gamma$: a full and faithful functor

$\Omega$: specialization

# The adjunction $\omega \dashv \iota$

The functor $\omega\colon \mathsf{Top} \longrightarrow \mathsf{V}\text{-}\mathsf{Top}$ maps a topological space $X$ to $\omega(X) := (X, \delta_X)$, where

$$\delta_X\colon X \times 2^X \longrightarrow [0,1], \quad \delta_X(x, A) = \begin{cases} 1 & x \in \overline{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Spaces of the form $\omega(X)$ are said to be topologically generated.

The right adjoint $\iota\colon \mathsf{V}\text{-}\mathsf{Top} \longrightarrow \mathsf{Top}$ of $\omega$ maps a V-topological space $(X, \delta)$ to the topological space $(X, \iota(\delta))$, of which a closed set is of the form $\lambda^{-1}(1)$ for some continuous function $\lambda\colon (X, \delta) \longrightarrow ([0,1], \delta_{\mathbb{K}})$.

The space $(X, \iota(\delta))$ is called the topological modification of $(X, \delta)$.

# The adjunction $\Gamma \dashv \Omega$

The functor $\Gamma\colon \text{V-Cat} \longrightarrow \text{V-Top}$ maps a V-category $(X,\alpha)$ to $\Gamma(X,\alpha) := (X,\Gamma(\alpha))$, where

$$\Gamma(\alpha)\colon X \times 2^X \longrightarrow [0,1], \quad \Gamma(\alpha)(x,A) = \begin{cases} 0 & A = \varnothing, \\ \sup\limits_{y\in A} \alpha(x,y) & A \neq \varnothing. \end{cases}$$

Spaces of the form $\Gamma(X,\alpha)$ are said to be Alexandroff.

The right adjoint $\Omega\colon \text{V-Top} \longrightarrow \text{V-Cat}$ of $\Gamma$ maps a V-topological space $(X,\delta)$ to the V-category $\Omega(X,\delta) := (X,\Omega(\delta))$, where $\Omega(\delta)(x,y) = \delta(x,\{y\})$.

The V-category $\Omega(X,\delta)$ is called the specialization of the space $(X,\delta)$.

## Example 5 (examples of closed sets of V-topological spaces)

- Let $X$ be a topological space. Then $\phi\colon X \longrightarrow [0,1]$ is a closed set of the topologically generated space $\omega(X)$ if and only if $\phi$ is upper semicontinuous in the usual sense.
- Let $X$ be a real-enriched category. Then $\phi\colon X \longrightarrow [0,1]$ is a closed set of the Alexandroff space $\Gamma(X)$ if and only if $\phi$ is a weight of $X$.

A weight of a V-category $X$ is a functor $\phi\colon X^{\mathrm{op}} \longrightarrow \mathsf{V}$. Explicitly, a weight of $X$ is a map $\phi\colon X \longrightarrow [0,1]$ such that $\phi(y) \,\&\, X(x,y) \le \phi(x)$ for all $x, y \in X$.

# Open ball topology

Let $X$ be a real-enriched category, $x \in X$ and $r < 1$. The open ball with center $x$ and radius $r$ is the set

$$B(x, r) := \{y \in X \mid X(x, y) > r\}.$$

The open ball topology of $X$ is the topology generated as a basis by its open balls.

### Proposition 6

For each real-enriched category $X$, the open ball topology of $X$ is the topological modification of its Alexandroff V-topology; that is,

$$\begin{tikzcd} \mathsf{V\text{-}Cat} \arrow[r, "\Gamma"] \arrow[dr, "\text{open ball}"'] & \mathsf{V\text{-}Top} \arrow[d, "\iota"] \\ & \mathsf{Top} \end{tikzcd}$$

# Scott-type topologies

To postulate Scott-type topologies of real-enriched categories, we need the concepts of forward Cauchy nets, Yoneda limits, formal balls, and some others.

## Definition 7 (forward Cauchy nets and Yoneda limits)

Suppose $\{x_i\}_{i\in D}$ is a net and $b$ is an element of a V-category $X$. We say that

(i) $\{x_i\}_{i\in D}$ is forward Cauchy if

$$\sup_{i\in D}\inf_{k\geq j\geq i} X(x_j, x_k) = 1.$$

(Enriched version of 'eventually monotone nets')

(ii) $b$ is a Yoneda limit of $\{x_i\}_{i\in D}$ if for all $y \in X$,

$$X(b, y) = \sup_{i\in D}\inf_{i\leq j} X(x_j, y).$$

(Enriched version of 'least eventual upper bound')

A real-enriched category is Yoneda complete if each of its forward Cauchy nets has a Yoneda limit.

A functor between real-enriched categories is Yoneda continuous if it preserves Yoneda limits of all forward Cauchy nets.

Then we have a category

$$\mathsf{V\text{-}Cat}^{\uparrow}$$

of V-categories and Yoneda continuous functors.

## Example 8

(i) If $\{a_i\}_{i\in D}$ is a forward Cauchy net of $\mathsf{V} = ([0,1], \alpha_L)$, then

  (a) $\sup_{i\in D}\inf_{j\geq i} a_j$ is a Yoneda limit of $\{a_i\}_{i\in D}$, so $\mathsf{V}$ is Yoneda complete;

  (b) $\{a_i\}_{i\in D}$ is order convergent; that is, $\sup_{i\in D}\inf_{j\geq i} a_j = \inf_{i\in D}\sup_{j\geq i} a_j$.

(ii) If $\{a_i\}_{i\in D}$ is a forward Cauchy net of $\mathsf{V}^{\mathrm{op}} = ([0,1], \alpha_R)$, then

  (a) $\inf_{i\in D}\sup_{j\geq i} a_j$ is a Yoneda limit of $\{a_i\}_{i\in D}$, so $\mathsf{V}^{\mathrm{op}}$ is Yoneda complete;

  (b) $\{a_i\}_{i\in D}$ is order convergent; that is, $\sup_{i\in D}\inf_{j\geq i} a_j = \inf_{i\in D}\sup_{j\geq i} a_j$.

## Definition 9 (g-Scott topology)

A subset $U$ of a real-enriched category $X$ is g-Scott open (generalized Scott open) if for every forward Cauchy net $\{x_i\}_{i\in D}$ and every Yoneda limit $x$ of $\{x_i\}_{i\in D}$, $x \in U$ implies that there exist $r < 1$ and $i \in D$ such that the open ball

$$B(x_j, r) := \{y \mid X(x_j, y) > r\}$$

is contained in $U$ whenever $j \geq i$.

The g-Scott open subsets of $X$ form a topology, called the g-Scott topology of $X$.

M.M. Bonsangue, F. van Breugel, J.J.M.M. Rutten, Generalized metric space: completion, topology, and powerdomains via the Yoneda embedding, Theoretical Computer Science 193 (1998) 1-51.

# The ordered set of formal balls

Suppose $X$ is a real-enriched category. A *formal ball* of $X$ is a pair $(x, r)$ with $x \in X$ and $r \in [0, 1]$, $x$ is called the center and $r$ the radius. For formal balls $(x, r)$ and $(y, s)$, let

$$(x, r) \sqsubseteq (y, s) \quad \text{if } r \leq s \,\&\, X(x, y).$$

The relation $\sqsubseteq$ is reflexive and transitive, hence an order.

The set of formal balls of $X$ ordered by $\sqsubseteq$ is denoted by $\mathrm{B}X$. Any formal ball $(x, 0)$ with radius 0 is a bottom element of $\mathrm{B}X$.

### Definition 10 (d-Scott topology)

The d-Scott topology of a real-enriched category $X$ is the topology inherited from the Scott topology of the ordered set $(\mathrm{B}X, \sqsubseteq)$ via the embedding

$$\eta_X : X \longrightarrow \mathrm{B}X, \quad x \mapsto (x, 1).$$

J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory, Cambridge University Press, Cambridge, 2013.

In the following we define the c-Scott topology via Scott V-topology.

### Definition 11 (Scott closed weight)

A weight $\lambda$ of a real-enriched category $(X, \alpha)$ is Scott closed if, as a functor $(X, \alpha) \longrightarrow \mathsf{V}^{\mathrm{op}}$, $\lambda$ is Yoneda continuous.

Scott closed weights of $(X, \alpha)$, as closed sets, define a V-topology $\Sigma(\alpha)$ on $X$, called the *Scott* V-*topology* of $(X, \alpha)$.

Assigning to each V-category its Scott V-topology defines a full and faithful functor

$$\Sigma\colon \mathsf{V\text{-}Cat}^{\uparrow} \longrightarrow \mathsf{V\text{-}Top}.$$

### Example 12

$([0,1], \delta_{\mathbb{K}}) = \Sigma([0,1], \alpha_R)$.

## Definition 13 (c-Scott topology)

The c-Scott topology of a real-enriched category $(X, \alpha)$ is the topological modification of its Scott V-topology; that is,

$$\begin{tikzcd} \mathsf{V}\text{-}\mathsf{Cat}^{\uparrow} \arrow[r, "\Sigma"] \arrow[dr, "\text{c-Scott}"'] & \mathsf{V}\text{-}\mathsf{Top} \arrow[d, "\iota"] \\ & \mathsf{Top} \end{tikzcd}$$

### Theorem 14

For each real-enriched category, g-Scott $\supseteq$ c-Scott $\supseteq$ d-Scott.

Let $X$ be a real-enriched category. The weights of $X$ (i.e., functors $X^{\mathrm{op}} \longrightarrow \mathsf{V}$) constitute a category $\mathcal{P}X$ with

$$\mathcal{P}X(\phi_1, \phi_2) = \inf_{x \in X} (\phi_1(x) \to \phi_2(x)).$$

An element $b$ of $X$ is called a colimit of a weight $\phi$ if for all $y \in X$,

$$X(b, y) = \mathcal{P}X(\phi, X(-, y)).$$

## Definition 15 (ideals of a V-category)

Let $X$ be a real-enriched category. A weight $\phi$ of $X$ is called an ideal if the functor

$$\mathcal{P}X(\phi, -)\colon \mathcal{P}X \longrightarrow \mathsf{V}$$

preserves (enriched) finite colimits, where $\mathsf{V} = ([0,1], \alpha_L)$.

✎ The terminology 'ideal' is chosen because an ideal of a ordered set (i.e., a directed lower set) is exactly a coprime in its ordered set of lower sets.

Relation between ideals, forward Cauchy nets, and directed subsets of formal balls:

## Proposition 16

For each weight $\phi$ of a real-enriched category $X$, the following are equivalent:

(1) $\phi$ is an ideal of $X$.

(2) $\sup_{x\in X}\phi(x)=1$ and $\mathrm{B}\phi:=\{(x,r)\in\mathrm{B}X\mid\phi(x)>r\}$ is a directed subset of $\mathrm{B}X$.

(3) $\phi=\sup_{i\in D}\inf_{j\geq i}X(-,x_j)$ for some forward Cauchy net $\{x_i\}_{i\in D}$ of $X$.

The ordered set $(\mathrm{B}\phi,\sqsubseteq)$ may be viewed as 'category of elements of $\phi$', ideals of $X$ as enriched version of 'ind-objects' or 'directed lower sets'.

# Two useful facts

Let $b$ be an element and $\{x_i\}_{i\in D}$ a forward Cauchy net of a V-category $X$; let

$$\phi = \sup_{i\in D}\inf_{j\geq i} X(-, x_j).$$

Then

- $b$ is a Yoneda limit of $\{x_i\}_{i\in D}$ if and only if $b$ is a colimit of the ideal $\phi$.
- For each weight $\lambda$ of $X$,

$$\mathcal{P}X(\phi, \lambda) = \inf_{i\in D}\sup_{j\geq i}\lambda(x_j) = \sup_{i\in D}\inf_{j\geq i}\lambda(x_j).$$

## Lemma 17 (characterizations of Scott closed weights)

For each weight $\lambda$ of a real-enriched category $X$, the following are equivalent:

(1) $\lambda$ is Scott closed.

(2) For every ideal $\phi$ of $X$, $\lambda(\operatorname{colim}\phi) \geq \mathcal{P}X(\phi, \lambda)$.

(3) For every forward Cauchy net $\{x_i\}_{i\in D}$ of $X$ and every Yoneda limit $x$ of $\{x_i\}_{i\in D}$,

$$\lambda(x) \geq \inf_{i\in D}\sup_{j\geq i}\lambda(x_j).$$

(4) For every directed subset $\{(x_i, r_i)\}_{i\in D}$ of formal balls with $\sup_i r_i = 1$ and every Yoneda limit $x$ of $\{x_i\}_{i\in D}$,

$$\lambda(x) \geq \inf_{i\in D}\sup_{j\geq i}\lambda(x_j).$$

# Proof of "c-Scott $\subseteq$ g-Scott"

We check that for each Scott closed weight $\lambda$ of $X$, the set $\{y \mid \lambda(y) < 1\}$ is g-Scott open. Let $\{x_i\}_{i \in D}$ be a forward Cauchy net and $x$ be a Yoneda limit of $\{x_i\}_{i \in D}$. We need to show that if $\lambda(x) = r < 1$, then there exist $s < 1$ and $i \in D$ such that the open ball $B(x_j, s)$ is contained in $U$ whenever $j \geq i$.

Since

$$\inf_{i \in D} \sup_{j \geq i} \lambda(x_j) \leq \lambda(x) < r = \inf_{\epsilon < 1} (\epsilon \to r),$$

there exist $\epsilon < 1$ and $i \in D$ such that $\epsilon \to r < 1$ and $\lambda(x_j) \leq \epsilon \to r$ whenever $j \geq i$. Pick $s$ with $\epsilon \to r < s < 1$. Then $s$ and $i$ satisfy the requirement.

# Proof of “d-Scott $\subseteq$ c-Scott”

Given a Scott closed set $F$ of the ordered set $(\mathrm{B}X, \sqsubseteq)$, define $\lambda_F : X \longrightarrow [0,1]$ by

$$\lambda_F(x) = \sup\{r \mid (x,r) \in F\}.$$

Then $\lambda_F$ is a Scott closed weight such that $\lambda_F^{-1}(1) = \eta_X(X) \cap F$, which shows that d-Scott topology is coarser than c-Scott topology.

To see that $\lambda_F$ is Scott closed, the characterization in Lemma 17 (4) is used.

## Proposition 18

Suppose & is a continuous Archimedean t-norm, i.e., & has no idempotent elements other than 0 and 1. Then for every standard real-enriched category, c-Scott = d-Scott.

## Proposition 18

Suppose & is a continuous Archimedean t-norm, i.e., & has no idempotent elements other than 0 and 1. Then for every standard real-enriched category, c-Scott = d-Scott.

✎ Suppose & is a continuous Archimedean t-norm. Then every Yoneda complete real-enriched category is standard.

✎ Suppose & is a continuous Archimedean t-norm. A real-enriched category $X$ is *standard* provided that for each directed subset $\{(x_i, r_i)\}_{i\in D}$ of $\mathrm{B}X$ with $\bigvee_{i\in D} r_i > 0$, if $\{(x_i, r_i)\}_{i\in D}$ has a join, then so do the following subsets:

(i) $\{(x_i, t \,\&\, r_i)\}_{i\in D}$ for each $t > 0$; and

(ii) $\{(x_i, t \to r_i)\}_{i\in D}$ for $t = \bigvee_{i\in D} r_i$.

## Proposition 19

For each continuous real-enriched category, g-Scott = c-Scott.

Let $X, Y$ be real-enriched categories, $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ be functors. We say that $f$ is left adjoint to $g$, or $g$ is right adjoint to $f$, and write $f \dashv g$, if

$$Y(f(x), y) = X(x, g(y))$$

for all $x \in X$ and $y \in Y$.

Let $X$ be a real-enriched category and let $\mathcal{J}X$ be the subcategory of $\mathcal{P}X$ composed of ideals that have colimits. Then

$$\mathrm{colim}\colon \mathcal{J}X \longrightarrow X$$

is left adjoint to the Yoneda embedding (with codomain restricted to $\mathcal{J}X$)

$$\mathrm{y}\colon X \longrightarrow \mathcal{J}X, \quad x \mapsto X(-,x).$$

If the functor $\mathrm{colim}\colon \mathcal{J}X \longrightarrow X$ has a left adjoint, we say that $X$ is *continuous*. In other words, $X$ is continuous if there exists a string of adjunctions

$$\twoheaddownarrow \dashv \mathrm{colim} \dashv \mathrm{y}\colon X \longrightarrow \mathcal{J}X.$$

## Sketch of the proof of Proposition 19.

Let $X$ be a continuous real-enriched category. For each $x \in X$ and $r < 1$, let

$$O(x, r) = \{y \in X \mid \Uparrow x(y) > r\},$$

where $\Uparrow x(y) = \Downarrow y(x)$. Then,

$$\{O(x, r) \mid x \in X, r < 1\}$$

is a basis for both the c-Scott topology and the g-Scott topology. $\square$

## Concluding remarks

- Topologies on real-enriched categories can be constructed with help of their 'natural V-topologies':

$$\begin{tikzcd} \mathsf{V}\text{-}\mathsf{Cat} \arrow[r, "?"] \arrow[dr, "\iota\circ ?"'] & \mathsf{V}\text{-}\mathsf{Top} \arrow[d, "\iota"] \\ & \mathsf{Top} \end{tikzcd}$$

  $? =$ enriched version of Scott topology, interval topology, Lawson topology, $\cdots$

- The related V-topologies themselves are important topological structures of real-enriched categories.

# Examples

Let $X$ be a real-enriched category.

(i) The interval V-topology of $X$ is generated by

$$\{X(-,x), X(x,-) \mid x \in X\}$$

as a subbasis for closed sets.

(ii) The Lawson V-topology of $X$ is generated by

$$\{X(x,-) \mid x \in X\} \cup \{\text{Scott closed weights}\}$$

as a subbasis for closed sets.

# Examples

(i) $\Gamma\colon \mathsf{V\text{-}Cat} \longrightarrow \mathsf{V\text{-}Top}$ is a full and faithful functor.

(ii) $\Sigma\colon \mathsf{V\text{-}Cat}^{\uparrow} \longrightarrow \mathsf{V\text{-}Top}$ is a full and faithful functor.

(iii) If $(X, \delta)$ is a sober V-topological space, then $\Omega(X, \delta)$ is Yoneda complete.

(iv) If $X$ is separated, Yoneda complete and continuous, then $\Sigma X$ is sober.

(v) A real-enriched category $X$ is Smyth complete (every forward Cauchy net converges in the open ball topology of its symmetrization) if, and only if, the V-topological space $\Gamma X$ is sober. In this case, $\Gamma X = \Sigma X$.