$\mathbb{N}$-compactness and $\mathbb{N}$-compact extensions in the absence of the Axiom of Choice

Eliza Wajch

Institute of Mathematics, University of Siedlce
Poland

The 38th Summer Conference on Topology and Its Applications
Coimbra, Portugal
July 8–12, 2024

Set-theoretic framework and some known weaker choice principles

- Zermelo-Fraenkel system $\mathbf{ZF}$.
- $\mathbf{CMC}$ (the axiom of countable multiple choice, Form 126 in [HR]): Every denumerable family of non-empty sets has a multiple choice function.
- $\mathbf{BPI}$ (the Boolean Prime Ideal Theorem, Form 14 in [HR]): Every Boolean algebra has a prime ideal.

Remark 1

$\mathbf{BPI}$ is equivalent in $\mathbf{ZF}$ to the statement: Every product of compact Hausdroff spaces is compact.

Remark 2

There are known models of $\mathbf{ZF}+\mathbf{BPI}+\neg\mathbf{CMC}$, $\mathbf{ZF}+\mathbf{CMC}+\neg\mathbf{BPI}$, $\mathbf{ZF}+\mathbf{BPI}+\mathbf{CMC}+\neg\mathbf{AC}$, $\mathbf{ZF}+\neg\mathbf{BPI}+\neg\mathbf{CMC}$.

Basic notation

- $\boldsymbol{X} = \langle X, \tau_X \rangle$, $\boldsymbol{Y} = \langle Y, \tau_Y \rangle$- topological spaces;
- for a set T, $T_{disc} := \langle T, \mathcal{P}(T) \rangle$- the discrete space with the underlying set T;
- $C(\boldsymbol{X}, \boldsymbol{Y})$- the set of all continuous mappings from $\boldsymbol{X}$ into $\boldsymbol{Y}$;
- $C(\boldsymbol{X}) := C(\boldsymbol{X}, \mathbb{R})$ where $\mathbb{R}$ is equipped with the natural topology; $C(\boldsymbol{X})$ is equipped with the topology of uniform convergence;
- for $f \in C(\boldsymbol{X})$, $Z(f) := f^{-1}[\{0\}]$ is the zero-set of f;
- $\mathcal{CO}(\boldsymbol{X})$ - the family of all clopen sets in $\boldsymbol{X}$;
- for $f \in C(\boldsymbol{X})$ and $\emptyset \neq A \subseteq X$,

$$\operatorname{osc}_A(f) := \sup\{|f(x) - f(y)|; x, y \in A\} \quad \text{and} \quad \operatorname{osc}_\emptyset(f) = 0.$$

Definitions of the ring $U_{\aleph_0}(\boldsymbol{X})$ and its special subrings

In what follows, we assume that $\boldsymbol{X}$ is a non-empty topological space. We focus on the following subrings of $C(\boldsymbol{X})$:

$$U_{\aleph_0}(\boldsymbol{X}) := \{f \in C(\boldsymbol{X}) : (\forall \epsilon \in (0, +\infty))(\exists \mathcal{A} \in [\mathcal{CO}(\boldsymbol{X})]^{\leq \omega}) \\ (\bigcup \mathcal{A} = X \quad \text{and} \quad (\forall A \in \mathcal{A})\, \mathrm{osc}_A(f) \leq \epsilon)\} \tag{1}$$

$$C_c(\boldsymbol{X}) := \{f \in C(\boldsymbol{X}) : |f[X]| \leq \aleph_0\} \tag{2}$$

$$C(\boldsymbol{X}, \mathbb{R}_{disc}) \quad \text{and} \quad C(\boldsymbol{X}, \mathbb{R}_{disc}) \cap C_c(\boldsymbol{X}) \tag{3}$$

Other subrings of $U_{\aleph_0}(\boldsymbol{X})$

More generally, we consider the collection $\mathcal{H}_{\aleph_0}(\boldsymbol{X})$ of all subrings $\mathbb{H}$ of $U_{\aleph_0}(\boldsymbol{X})$ satisfying the following conditions:

(i) $C(\boldsymbol{X}, \mathbb{R}_{disc}) \cap C_c(\boldsymbol{X}) \subseteq \mathbb{H}$;

(ii) $(\forall h \in \mathbb{H})(0 \leq h \to \sqrt{h} \in \mathbb{H})$.

The ring $C(\boldsymbol{X})$ is equipped with the topology induced by the metric of uniform convergence ρ_u defined as follows:

$$(\forall f, g \in C(\boldsymbol{X}))\rho_u(f, g) := \sup\{\min\{|f(x) - g(x)|, 1\} : x \in X\}.$$

Theorem 1

$[\boldsymbol{ZF} + \boldsymbol{CMC}]$ $(\forall\ \mathbb{H} \in \mathcal{H}_{\aleph_0}(\boldsymbol{X}))\, \mathrm{cl}_{\rho_u}(\mathbb{H}) = U_{\aleph_0}(\boldsymbol{X})$.

What has made us interested in $U_{\aleph_0}(X)$?

Theorem 2

(Keremedis, Olfati, Wajch, 2023) $[\boldsymbol{ZF}+\boldsymbol{CMC}]$

(i) *A completely regular space $\boldsymbol{X}$ is strongly zero-dimensional if and only if $U_{\aleph_0}(\boldsymbol{X}) = C(\boldsymbol{X})$.*

(ii) *A zero-dimensional space $\boldsymbol{X}$ is a P-space if and only if $U_{\aleph_0}(\boldsymbol{X}) = C(\boldsymbol{X}, \mathbb{R}_{disc}) = C(\boldsymbol{X})$.*

Theorem 3

(Olfati, Wajch, 2022) $[\boldsymbol{ZF}+\boldsymbol{CMC}]$ *For every Tychonoff space $\boldsymbol{X}$, the following are equivalent:*

(i) *$\boldsymbol{X}$ is strongly zero-dimensional;*

(ii) $C(\boldsymbol{X}) = \{f \upharpoonright X : f \in U_{\aleph_0}(v\boldsymbol{X})\}$;

(iii) *$\boldsymbol{X}$ is zero-dimensional and $C(\boldsymbol{X}) = \{f \upharpoonright X : f \in U_{\aleph_0}(v_0\boldsymbol{X})\}$.*

Definitions of characters and real ideals of subrings of $C(\boldsymbol{X})$

Let $\mathbb{H}$ be a subring of $C(\boldsymbol{X})$ which contains all constant functions from $C(\boldsymbol{X})$. For $c \in \mathbb{R}$, c stands also for the constant function on X having the unique value c.

(a) A *character* on $\mathbb{H}$ is a function $\chi : \mathbb{H} \to \mathbb{R}$ which satisfies the following conditions:

  (i) $(\forall f, g \in \mathbb{H})\ \chi(f+g) = \chi(f) + \chi(g)$;

  (ii) $(\forall f, g \in \mathbb{H})\ \chi(f \cdot g) = \chi(f) \cdot \chi(g)$;

  (iii) $(\forall c \in \mathbb{R})\ \chi(c) = c$.

(b) For $w \in X$, the *character* on $\mathbb{H}$ *determined by* w (or, the *evaluation* on $\mathbb{H}$ at w) is the function $\chi_w : \mathbb{H} \to \mathbb{R}$ defined as follows:

$$(\forall f \in \mathbb{H})\ \chi_w(f) = f(w).$$

(c) An ideal M of $\mathbb{H}$ is called a *real ideal* of $\mathbb{H}$ if the quotient ring $\mathbb{H}/M$ is isomorphic with the field $\mathbb{R}$, and M is *fixed* if there exists $p \in X$ such that $M = \{f \in \mathbb{H} : f(p) = 0\}$.

Older theorems about characters on $C(\boldsymbol{X})$ and $C_c(\boldsymbol{X})$

Theorem 4

(Shirota 1952, Boulabiar 2014) $[\boldsymbol{ZF}]$ *A non-empty Tychonoff space* $\boldsymbol{X}$ *is realcompact if and only if every character on* $C(\boldsymbol{X})$ *is an evaluation on* $C(\boldsymbol{X})$ *at a point of* $\boldsymbol{X}$.

Theorem 5

(Olfati 2016) $[\boldsymbol{ZFC}]$ *Let* $\boldsymbol{X}$ *be an* $\mathbb{N}$*-compact space and let* χ *be a character on* $C_c(\boldsymbol{X})$. *Then there exists a unique* $w \in X$ *such that, for every* $f \in C_c(\boldsymbol{X})$, $\chi(f) = f(w)$.

New results about characters

Theorem 6

(Olfati, Wajch 2024) $[\boldsymbol{ZF}]$ *Let* $\boldsymbol{X}$ *be a non-empty* $\mathbb{N}$*-compact space and let* χ *be a character on a subring* $\mathbb{H}$ *of* $C(\boldsymbol{X})$. *Then the following conditions are satisfied:*

(i) *if* $\mathbb{H} = C(\boldsymbol{X}, \mathbb{R}_{disc})$ *or* $\mathbb{H} = C_c(\boldsymbol{X})$, *or* $\mathbb{H} = C(\boldsymbol{X}, \mathbb{R}_{disc}) \cap C_c(\boldsymbol{X})$, *then there exists a unique* $w \in X$ *such that, for every* $h \in \mathbb{H}$, $\chi(h) = h(w)$;

(ii) *if* $\mathbb{H} \in \mathcal{H}_{\aleph_0}(\boldsymbol{x})$, *in particular, if* $\mathbb{H} = U_{\aleph_0}(\boldsymbol{X})$, *then* $\boldsymbol{CMC}$ *implies that there exists a unique* $w \in X$ *such that, for every* $h \in \mathbb{H}$, $\chi(h) = h(w)$.

Remark 3

For a non-empty zero-dimensional T_1-space $\boldsymbol{X}$, the assumption that $\boldsymbol{X}$ is $\mathbb{N}$-compact is essential in Theorem 6.

A characterization of $\mathbb{N}$-compactness via real ideals

Theorem 6 is applied to the proof of the following theorem:

Theorem 7

(Olfati, Wajch 2024) [$\mathbf{ZF}$] Let $\mathbf{X}$ be a non-empty zero-dimensional T_1-space. Then the following conditions are equivalent:

(i) *$\mathbf{X}$ is $\mathbb{N}$-compact;*

(ii) *every real ideal of $C(\mathbf{X}, \mathbb{R}_{disc})$ is fixed;*

(iii) *every real ideal of $C_c(\mathbf{X})$ is fixed;*

(iv) *every real ideal of $C(\mathbf{X}, \mathbb{R}_{disc}) \cap C_c(\mathbf{X})$ is fixed.*

Furthermore, $\mathbf{CMC}$ implies that each of the conditions (i)–(iv) is equivalent to the following condition:

(v) *every real ideal of $U_{\aleph_0}(\mathbf{X})$ is fixed.*

More applications of characters

Among other results, Theorem 6 is applied to the proof of the following theorem:

Theorem 8

(Olfati, Wajch 2024) $[\boldsymbol{ZF} + \boldsymbol{CMC}]$ *Let* $\boldsymbol{X}$ *be a non-empty* $\mathbb{N}$*-compact space which admits its Banaschewski compactification* $\beta_0\boldsymbol{X}$. *Suppose that* $\mathbb{H}$ *is a subring of* $U_{\aleph_0}(\boldsymbol{X})$ *such that* $C_c(\boldsymbol{X}) \subseteq \mathbb{H}$ *and, for every* $h \in \mathbb{H}$, *if* $Z(h) = \emptyset$, *then* $\frac{1}{h} \in \mathbb{H}$, *for every* $h \in \mathbb{H}$ *with* $0 \leq h$, *we have* $\sqrt{h} \in \mathbb{H}$.
Let $\boldsymbol{Y}$ *be a realcompact space such that the rings* $\mathbb{H}$ *and* $C(\boldsymbol{Y})$ *are isomorphic. Then the spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are homeomorphic and strongly zero-dimensional. Furthermore, if* $\mathbb{H} = C_c(\boldsymbol{X})$, *then the spaces* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are both functionally countable.*

Corollaries

Corollary 1

$[\boldsymbol{ZF}+\boldsymbol{CMC}]$ Let $\boldsymbol{X}$ be a non-empty strongly zero-dimensional T_1-space which admits its Banaschewski compactification. Then there exists a Tychonoff space $\boldsymbol{Y}$ such that the rings $C_c(\boldsymbol{X})$ and $C(\boldsymbol{Y})$ are isomorphic if and only if $\boldsymbol{X}$ is functionally countable.

Corollary 2

$[\boldsymbol{ZF}+\boldsymbol{CMC}]$ Let $\boldsymbol{X}$ be a non-empty $\mathbb{N}$-compact space which admits its Banaschewski compactification. Then $\boldsymbol{X}$ is strongly zero-dimensional if and only if there exists a Tychonoff space $\boldsymbol{Y}$ such that the rings $U_{\aleph_0}(\boldsymbol{X})$ and $C(\boldsymbol{Y})$ are isomorphic.

On Banaschewski compactifications

For a non-empty space $\boldsymbol{X}$ and a subring $\mathbb{H}$ of $C(\boldsymbol{X})$, $\boldsymbol{Max}(\mathbb{H})$ is the set $Max(\mathbb{H})$ of all maximal ideals of $\mathbb{H}$ equipped with the *hull-kernel topology*. The base of $\boldsymbol{Max}(\mathbb{H})$ is the family

$$\mathcal{B} = \{\{M \in Max(\mathbb{H}) : f \notin M\} : f \in \mathbb{H}\}.$$

Theorem 9

(Wajch 2024) $[\boldsymbol{ZF} + \boldsymbol{CMC}]$ *Let* $\boldsymbol{X}$ *be a non-empty zero-dimensional* T_1*-space. Let* $\mathbb{H}$ *be a subring of* $U_{\aleph_0}(\boldsymbol{X})$ *such that* $C_c(\boldsymbol{X}) \subseteq \mathbb{H}$ *and, for every* $h \in \mathbb{H}$*, if* $Z(h) = \emptyset$*, then* $\frac{1}{h} \in \mathbb{H}$*. Then the Banaschewski compactification* $\beta_0\boldsymbol{X}$ *exists if and only if* $\boldsymbol{Max}(\mathbb{H})$ *is compact. Furthermore, if* $\beta_0\boldsymbol{X}$ *exists, then* $\beta_0\boldsymbol{X} \approx \boldsymbol{Max}(\mathbb{H})$.

Some equivalences of $\boldsymbol{BPI}$

Theorem 10

$[\boldsymbol{ZF} + \boldsymbol{CMC}]$ *The following are equivalent:*

(i) $\boldsymbol{BPI}$;

(ii) *every zero-dimensional T_1-space admits its Banaschewski compactification;*

(iii) *for every zero-dimesional T_1-space $\boldsymbol{X}$, the space $\boldsymbol{Max}(U_{\aleph_0}(\boldsymbol{X}))$ is compact;*

(iv) *for every zero-dimesional T_1-space $\boldsymbol{X}$, the space $\boldsymbol{Max}(C_c(\boldsymbol{X}))$ is compact.*

Moreover, in (ii)-(iv) "zero-dimensional T_1-space" can be replaced with "Cantor cube".

On Herrlich and Chew theorem

Theorem 11

(Herrlich (1967), independently Chew (1970)) [$\boldsymbol{ZFC}$] *A zero-dimensional T_1-space is $\mathbb{N}$-compact if and only if every ultrafilter in $\mathcal{CO}(\boldsymbol{X})$ with c.i.p. is fixed.*

Theorem 12

(Olfati, Wajch 2022-23) The Herrlich-Chew Theorem is valid in $\boldsymbol{ZF}$.

Theorem 13

(Olfati, Wajch 2022-23) [$\boldsymbol{ZF} + \boldsymbol{CMC}$] *A zero-dimensional T_1-space is $\mathbb{N}$-compact if and only if every ultrafilter in $\mathcal{CO}_\delta(\boldsymbol{X})$ with c.i.p is fixed where*

$$\mathcal{CO}_\delta(\boldsymbol{X}) = \{\bigcap \mathcal{A} : \mathcal{A} \in [\mathcal{CO}]^{\leq\omega} \setminus \{\emptyset\}\}.$$

On Hewitt's characterization of realcompactness

Theorem 14

*(Hewitt, 1948) [**ZFC**] A Tychonoff space is realcompact if and only if every z-ultrafilter in **X** with c.i.p. is fixed.*

Definition

*Let **X** be a topological space. A family $\mathcal{A}$ of zero-sets of **X** is functionally accessible if there exists a subfamily $\{g_Z : Z \in \mathcal{A}\}$ of $C(\boldsymbol{X})$ such that, for every $Z \in \mathcal{A}$, $Z = Z(g_Z)$. A z-filter $\mathcal{F}$ in **X** has the weak countable intersection property if, for every denumerable functionally accessible subfamily $\mathcal{A}$ of $\mathcal{F}$, $\bigcap \mathcal{A} \neq \emptyset$.*

Theorem 15

*(Olfati, Wajch, 2022-24) [**ZF**] A Tychonoff space **X** is realcompact if and only if every z-ultrafilter in **X** with the weak countable intersection property is fixed.*

Basic new references

- K. Keremedis, A. R. Olfati and E. Wajch: *On P-spaces and G_δ-sets in the absence of the Axiom of Choice*, Bull. Belg. Math. Soc. Simon Stevin 30 (2023), 194–236.
- A. R. Olfati and E. Wajch: ***E**-compact extensions in the absence of the Axiom of Choice*, preprint, (2022-23) arXiv:2211.00411
- A. R. Olfati and E. Wajch: *Banaschewski compactifications via special rings of functions in the absence of the axiom of choice*, submitted manuscript.
- A. R. Olfati and E. Wajch: *Characterizations of $\mathbb{N}$-compactness and realcompactness via ultrafilters in the absence of the axiom of choice*, a new manuscript.

Some basic older references

- K. Boulabiar: *Characters on $C(X)$*, Can. Math. Bull. 58(1) (2015), 7–8.
- [HR] P. Howard and J. E. Rubin: Consequences of the axiom of choice. Math. Surveys and Monographs 59, A.M.S., Providence R.I. 1998.
- K. Keremedis and E. Wajch: *Hausdorff compactifications in* $\boldsymbol{ZF}$, Topology Appl. 258 (2019), 79–99.
- A. R. Olfati: *On a question of $C_c(X)$*, Commentat. Math. Univ. Carol. 57 (2) (2016), 253–260.
- T. Shirota: *A class of topological spaces*, Osaka Math. J. 4, No. 1 (1952), 23–40.

Other older references

K. P. Chew: *A characterization of $\mathbb{N}$-compact spaces*, Proc. Am. Math. Soc. 26 (1970), 679–682.

H. Herrlich, $\mathfrak{E}$-Compacte Räume, Math. Zeitschr. 96 (1967), 228–255.

E. Hewitt: *Rings of continuous real-valued functions. I*, Trans. Am. Math. Soc. 64 (1948), 45–99.

… and many other papers or books.

Thank you for your attention very much!