On the Localization of Antisymmetric T_0-Quasi-Metric Spaces

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Basic Definitions

Let X be a set and $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

T_0-quasi-metrics:

A quasi-pseudometric d is called *T_0-quasi-metric* provided that d also satisfies the condition:
$d(x, y) = 0 = d(y, x)$ implies that $x = y$, for each $x, y \in X$,

Conjugate quasi-pseudometric

Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* d.

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A quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric* .

Conjugate quasi-pseudometric

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In addition, *symmetrization metric* d^s is described as $d^s = d \vee d^{-1}$.

Symmetric Connectedness

Let (X, d) be a T_0-quasi-metric space. A pair $(x, y) \in X \times X$ will be called *symmetric* if $d(x, y) = d(y, x)$.

Symmetric Connectedness

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A finite sequence of points in X, starting at x and ending with y, is called a (finite) *symmetric path* $P_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ (where $n \in \mathbb{N}$) from x to y provided that all the pairs (x_i, x_{i+1}) are symmetric ($i \in \{0, \ldots, n-1\}$).

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For any $x \in X$ the path $P_{x,x} = (x, x)$ or the pair (x, x) will be called a *loop*.

Standard T_0-quasi-metric

Example

On the set $\mathbb{R}$ of the reals, let us take $u(x, y) = (x - y) \vee 0$ whenever $x, y \in \mathbb{R}$. Then u is called the *standard* T_0-quasi-metric on $\mathbb{R}$. Observe that in $(\mathbb{R}, u)$ the only symmetric pairs are *trivial*, that is, are the loops.

Symmetric Connectedness

Definition

$x \in X$ is called *symmetrically connected* to $y \in X$ if there is a symmetric path $P_{x,y}$, starting at the point x and ending at the point y.

Comments

It is useful to assume that no point occurs twice in a path $P_{x,y}$, except that possibly $x = y$. (So our paths will be simple, but can be closed.)

We note that "symmetrically connected" is an equivalence relation on the set of all points of X. The equivalence class of a point $x \in X$ will be called the *symmetry component* of x.

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Symmetrically connected space:

A T_0-quasi-metric space (X, d) such that all the equivalence classes of points in X agree with X is called *symmetrically connected*.

Some symmetric structures

For a T_0-quasi-metric space (X, d), the set

$$Z_d = \{(x, y) \in X \times X : d(x, y) = d(y, x)\}$$

is called *the set of symmetric pairs* of (X, d). (Mostly, it will suffice to write Z instead of Z_d.)

Clearly, the relation $Z_d = Z$ is reflexive and symmetric.

For a symmetric pair (x, y) in (X, d), that is for $(x, y) \in Z$, we have that

$$d^s(x, y) = d(x, y) = d^{-1}(x, y).$$

Symmetric connectedness

Let (X, d) be a T_0-quasi-metric space. For $x \in X$ we shall use the notation

$$Z(x) = \{y \in X : (x, y) \in Z\}$$

and call $Z(x)$ the *symmetry set of* x.

Furthermore C_d (or briefly C) will denote the *symmetric connectedness relation*. So, clearly $Z \subseteq C$ and for each $x \in X$ we have:

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Hence $C(x)$ is the symmetry component of $x \in X$.

Some observations

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(d) Each metric space is symmetrically connected.

Antisymmetric T_0-quasi-metrics

A T_0-quasi-metric space (X, d) is called *antisymmetric* if

$$Z_d = \{(x, x) : x \in X\}$$

equals to the *diagonal* Δ_X of X (equivalently, each symmetry component of (X, d) is a singleton, that is $C_d(x) = \{x\})$.

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Observe that $(\mathbb{R}, u)$ is such a T_0-quasi-metric space.

• "Metric" and "antisymmetric" are in some sense properties opposite to each other.

Some observations on antisymmetry

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(d) An antisymmetric T_0-quasi-metric space with at least two points is not symmetrically connected.

- In particular the T_0-quasi-metric space $(\mathbb{R}, u)$ is not symmetrically connected.

Illustrating example

Let $X = \mathbb{R}^3$ be equipped with the T_0-quasi-metric d defined by

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = (x_1 - y_1) \vee (x_2 - y_2) \vee (x_3 - y_3) \vee 0.$$

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Fix $x > 0$. A straightforward calculation reveals that both $((0,0,0),(x,-x,x))$ and $((x,-x,x),(0,0,x))$ are symmetric pairs. Thus $(\mathbb{R}^3, d)$ is not antisymmetric space.

On the other hand, $((0,0,0),(0,0,x))$ is not a symmetric pair. So we have the fact that

$$Z((0,0,0)) \subsetneq C((0,0,0))$$

for the space $(\mathbb{R}^3, d)$.

Illustrating example

In particular $C((0,0,0))$ is not a metric subspace of $(\mathbb{R}^3, d)$, since the triangle determined by the points $(0,0,0), (x,-x,x), (0,0,x)$ has one side ($=$ pair) which is not symmetric.

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In particular $C((0,0,0))$ is not a metric subspace of $(\mathbb{R}^3, d)$, since the triangle determined by the points $(0,0,0), (x,-x,x), (0,0,x)$ has one side ($=$ pair) which is not symmetric.
Specifically, $C((0,0,0)) = \mathbb{R}^3$ and so, $(\mathbb{R}^3, d)$ is symmetrically connected.

Symmetry graph

For a T_0-quasi-metric space (X, d), we define the *symmetry graph* (G, K) of (X, d) as:

symmetry graph:

The set G of vertices of the symmetry graph equals to the set X and the set K of edges of G is defined as;
$\{x, y\}$ is an edge of G if and only if $(x, y) \in Z_d$., for $x, y \in X$.

Relations with "Connectedness" in graph theory

A (possibly infinite) graph is called *connected* if any two vertices x and y can be connected by a (finite, that is, having finitely many steps) path.

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Hence

Theorem

A T_0-quasi-metric space (X, d) is symmetrically connected if and only if the symmetry graph (G, K) is connected in the sense of graph theory.

Antisymmetric pairs

Given a T_0-quasi-metric space (X, d), we can obviously also study the set $R_d := (X \times X) \setminus Z_d$, that is, the set of what we call the *antisymmetric pairs* of (X, d).

Antisymmetric pairs

The investigation of R_d corresponds to a study of the complementary graph $(\overline{G}, \overline{K})$ of the *symmetry graph* (G, K) of (X, d).

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Note here that the vertex set G of a graph (G, K) and the vertex set $\overline{G}$ of its complementary graph $(\overline{G}, \overline{K})$ are equal and an edge belongs to the edge set $\overline{K}$ of $\overline{G}$ if and only if it is missing in the edge set K of G.

Antisymmetric Connectedness

In a T_0-quasi-metric space (X, d), two points $x, y \in X$ will be called *antisymmetrically connected* if there is a path $P_{x,y} = (x_0, \ldots, x_n)$ with $n \in \mathbb{N}$, $x_0 = x$ and $x_n = y$ such that each pair (x_i, x_{i+1}) with $i \in \{0, \ldots, n-1\}$ is an antisymmetric pair, that is $d(x, y) \neq d(y, x)$, or a loop of (X, d).

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Antisymmetrically connected space:

Let $T_d := \{(x,y) \in X \times X : x$ and y are antisymmetrically connected in $(X,d)\}$.
Hence

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Antisymmetrically connected space:

Let $T_d := \{(x, y) \in X \times X : x \text{ and } y \text{ are antisymmetrically connected in } (X, d)\}$.
Hence if $T_d = X \times X$, then the T_0-quasi-metric space (X, d) will be called *antisymmetrically connected*.

Antisymmetric Connectedness

Obviously $T_d := \{(x, y) \in X \times X : x$ and y are antisymmetrically connected in $(X, d)\}$ is an equivalence relation on X.
• The equivalence classes $T_d(x)$ of the points $x \in X$, will be called *antisymmetry components*.

Some observations on antisymmetric connectedness

(a) A T_0-quasi-metric space is metric if and only if all its antisymmetry components are singletons.

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(b) Any T_0-quasi-metric space that is antisymmetric is antisymmetrically connected.

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(a) A T_0-quasi-metric space is metric if and only if all its antisymmetry components are singletons.

(b) Any T_0-quasi-metric space that is antisymmetric is antisymmetrically connected.

(c) A T_0-quasi-metric space (X, d) is antisymmetrically connected if and only if the complementary graph $(\overline{G}, \overline{K})$ of the symmetry graph (G, K) of (X, d) is connected.

More on symmetry graphs of T_0-quasi-metric spaces

Let Z be a reflexive and symmetric relation on a set X. In this case;
- there is a T_0-quasi-metric d on X such that $Z = Z_d$.

- Any symmetric irreflexive binary relation R on a set X is the set of antisymmetric pairs of a T_0-quasi-metric space.

Hence

Corollary

Any graph with vertex set X is the symmetry graph for some T_0-quasi-metric on X.

Further relationships with graph theory

Let (X, d) be a T_0-quasi-metric space. Then (X, d) is symmetrically connected or antisymmetrically connected.

Of course, this is the result of the fact "For any graph G, G is connected or the complement of G is connected." in the sense of graph theory.

Sorgenfrey line as illustrating example

On the set $\mathbb{R}$ of the reals, set $s(x,y) = \min\{x-y, 1\}$ if $x \geq y$, and $s(x,y) = 1$ if $x < y$. Then $(\mathbb{R}, s)$ is the so-called (bounded) T_0-quasi-metric Sorgenfrey line.
Note that s is not a metric and it generates "Sorgenfrey Topology". Also, the symmetrization metric s^s is discrete metric.

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Note that s is not a metric and it generates "Sorgenfrey Topology". Also, the symmetrization metric s^s is discrete metric.
Here, clearly the space $(\mathbb{R}, s)$ is symmetrically connected as well as antisymmetrically connected.

Sorgenfrey line as illustrating example

Let $x, y \in \mathbb{R}$ and $x < y$. Then

- $(x, y + 1, y)$ is a path from x to y consisting of symmetric pairs.

Moreover, now if we choose $n \in \mathbb{R}$ such that $\frac{1}{n}(y - x) < 1$. Then

Sorgenfrey line as illustrating example

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Moreover, now if we choose $n \in \mathbb{R}$ such that $\frac{1}{n}(y-x) < 1$. Then

- $(x, x + \frac{(y-x)}{n}, x + 2\frac{(y-x)}{n}, \ldots, x + \frac{(n-1)(y-x)}{n}, y)$ is a path from x to y consisting of antisymmetric pairs.

Locally Antisymmetric T_0-Quasi-Metric Spaces

Definition

Let (X, d) be a T_0-quasi-metric space. If every point $x \in X$ has a τ_{d^s}-neighborhood U such that d is an antisymmetric T_0-quasi-metric on U, then (X, d) is called *locally mantisymmetric space*.

Antisymmetric $\longleftrightarrow$ Locally Antisymmetric

Proposition

Any antisymmetric T_0-quasi-metric space is locally antisymmetric.

The converse of it may not be true:

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Example

Consider the T_0-quasi-metric product space $[-\frac{1}{4}, \frac{1}{4}] \times \{0, 1\}$ with the usual sup product T_0-quasi-metric D as

$$D((x, y), (a, b)) = u(x, a) \vee q(y, b)$$

where u is the standard (restricted) T_0-quasi-metric on $[-\frac{1}{4}, \frac{1}{4}]$ and q is the discrete metric on $\{0, 1\}$.

Some Observations on Locally Antisymmetric Spaces

Proposition

Let (X, d) be a T_0-quasi-metric space. Then (X, d) is locally antisymmetric if and only if the conjugate space (X, d^{-1}) is locally antisymmetric.

Some Observations on Locally Antisymmetric Spaces

Theorem

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. If (X, d) is locally antisymmetric space then (A, d_A) is locally antisymmetric subspace.

Some Observations on Locally Antisymmetric Spaces

Theorem

Let (X, d), (Y, q) be T_0-quasi-metric spaces and $f : X \to Y$ be a surjective isometry. In this case,

(X, d) is locally antisymmetric if and only if (Y, q) is locally antisymmetri

Some Observations on Locally Antisymmetric Spaces

Proposition

Let (X, d) be a T_0-quasi-metric space. If the T_0-quasi-metric subspace (A, d_A) is locally antisymmetric and $B \subseteq X$ is τ_{d^s}-open then the subspace $(A \cap B, d_{A \cap B})$ is locally antisymmetric.

Some Observations on Locally Antisymmetric Spaces

Remark

Let (X, d), (Y, q) be locally antisymmetric T_0-quasi-metric spaces, and the function D is described as

$$D((x, y), (a, b)) = d(x, a) \vee q(y, b)$$

on the product set $X \times Y$. In this case, the product T_0-quasi-metric space $(X \times Y, D)$ may not be locally antisymmetric.

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For example;

Some Observations on Locally Antisymmetric Spaces

Example

Let $X = \mathbb{R}^2$ be equipped with the T_0-quasi-metric D defined by

$$D((x_1, x_2), (y_1, y_2)) = (x_1 - y_1) \vee (x_2 - y_2) \vee 0$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

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where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.
Here, $(\mathbb{R}^2, D) = (\mathbb{R}, u) \times (\mathbb{R}, u)$, that is
$D((x_1, x_2), (y_1, y_2)) = u(x_1, y_1) \vee u(x_2, y_2)$ where
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$D((x_1, x_2), (y_1, y_2)) = u(x_1, y_1) \vee u(x_2, y_2)$ where
$u(a, b) = (a - b) \vee 0$.

Note that the space $(\mathbb{R}, u)$ is locally antisymmetric since it is antisymmetric.

Some Observations on Locally Antisymmetric Spaces

But the product space $(\mathbb{R}^2, D)$ is not locally antisymmetric: the symmetrization topology τ_{D^s} is the usual Euclidean topology on $\mathbb{R}^2$. So each neighborhood of point $(0,0)$ contains a usual 2ϵ-open ball, and the pair $((x,-x),(-x,x))$ for $0 < x < \epsilon$ is symmetric pair in that ball.

Some Observations on Locally Antisymmetric Spaces

Corollary

If (X, d) is a finite T_0-quasi-metric space then (X, d) is locally antisymmetric.

Antisymmetrically connected $\stackrel{?}{\longleftrightarrow}$ Locally antisymmetric

Generally, there is no any relation between antisymmetric connectedness and local antisymmetricness as you will see in the next examples.

However, we will present a corollary with the help of a specific condition in the next section.

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However, we will present a corollary with the help of a specific condition in the next section.

Example

Let $X = \mathbb{R}^2$ be equipped with the T_0-quasi-metric d defined by

$$d((x_1, x_2), (y_1, y_2)) = (x_1 - y_1) \vee (x_2 - y_2) \vee 0$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Thus, the T_0-quasi-metric space $(\mathbb{R}^2, d)$ is antisymmetrically connected. But it is well-known that $(\mathbb{R}^2, d)$ is not locally antisymmetric.

Antisymmetrically connected $\overset{?}{\longleftrightarrow}$ Locally antisymmetric

Example

Consider the discrete metric space (X, d). It is clear that (X, d) is not antisymmetrically connected since it contains symmetric points. On the other hand, it is easy to show that (X, d) is locally antisymmetric. Indeed, $d^s = d$ and the singleton sets $\{x\}$ for $x \in X$ are $\tau_d = \tau_{d^s}$-open, that is τ_{d^s}-neighborhoods of x. Also, the sets $\{x\}$ are antisymmetric w.r.t d.

Antisymmetrically connected $\longleftrightarrow^?$ Locally antisymmetric

Remark

Any T_0-quasi-metric space may be both locally antisymmetric and antisymmetrically connected space:

Example

Take any antisymmetric space such as $(\mathbb{R}, u)$, then it will be directly locally antisymmetric, and antisymmetrically connected.

Antisymmetrically connected $\longleftrightarrow^?$ Locally antisymmetric

Remark

Any T_0-quasi-metric space may be neither locally antisymmetric nor antisymmetrically connected space:

Example

Consider a non-discrete metric space (X, q) with at least 2-points. Then $q^s = q$ and for a non-discrete point $x \in X$, the τ_{q^s}-neighborhood N_x of x cannot be antisymmetric. Thus, (X, q) is not locally antisymmetric. Also it is not antisymmetrically connected since it is a metric space.

Local Antisymmetricness $\stackrel{?}{\longleftrightarrow}$ Local Antisymmetric Connectedness

Definition

Let (X, d) be a T_0-quasi-metric space. If for every point $x \in X$, the antisymmetry component $T_d(x)$ is τ_{d^s}-open then (X, d) is called *locally antisymmetrically connected space*.

Local Antisymmetricness $\overset{?}{\longleftrightarrow}$ Local Antisymmetric Connectedness

Proposition (Main Motivation)

A locally antisymmetric T_0-quasi-metric space is locally antisymmetrically connected.

Local Antisymmetricness $\overset{?}{\longleftrightarrow}$ Local Antisymmetric Connectedness

CounterExample

Recall the T_0-quasi-metric space $(\mathbb{R}^3, d)$ where
$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = (x_1 - y_1) \vee (x_2 - y_2) \vee (x_3 - y_3) \vee 0.$

The space $(\mathbb{R}^3, d)$ is locally antisymmetrically connected since it is antisymmetrically connected. But it is not locally antisymmetric: the symmetrization topology is the usual Euclidean topology on $\mathbb{R}^3$. So each neighborhood of point $(0, 0, 0)$ contains a usual 2ϵ-open ball, and the pair $((x, -x, 0), (-x, x, 0))$ for $0 < x < \epsilon$ is symmetric pair in that ball.

Local Antisymmetricness $\overset{?}{\longleftrightarrow}$ Local Antisymmetric Connectedness

The converse of proposition will be true under a condition, and finally, we have a characterization as follows:

Theorem

Let (X, d) be a T_0-quasi-metric space and the relation $((X \times X) \setminus Z_d) \cup \triangle$ be transitive. In this case, (X, d) is locally antisymmetrically connected space if and only if (X, d) is locally antisymmetric space.

Local Antisymmetricness $\overset{?}{\longleftrightarrow}$ Antisymmetric Connectedness

Hence, the next result will be obvious via the main motivation:

Corollary

If (X, d) is a locally antisymmetric space and the symmetrization topology τ_{d^s} is connected then the space (X, d) is antisymmetrically connected.

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THANK YOU!