An uncountable family of smooth fans that admit transitive homeomorphisms

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Basic definitions and notations

Definition

A continuum is a non-empty compact connected metric space. A subcontinuum is a subspace of a continuum, which is itself a continuum.

Definition

Let X be a continuum.

1. The continuum X is unicoherent, if for any subcontinua A and B of X such that $X = A \cup B$, the compactum $A \cap B$ is connected.
2. The continuum X is hereditarily unicoherent provided that each of its subcontinua is unicoherent.
3. The continuum X is a dendroid, if it is an arcwise connected, hereditarily unicoherent continuum.
4. Let X be a continuum. If X is homeomorphic to $[0, 1]$, then X is an arc.
5. A point x in an arc X is called an end-point of the arc X, if there is a homeomorphism $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$.
6. Let X be a dendroid. A point $x \in X$ is called an end-point of the dendroid X, if for every arc A in X that contains x, x is an end-point of A. The set of all end-points of X will be denoted by $E(X)$.
7. A continuum X is a simple triod, if it is homeomorphic to $([-1, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$.

Basic definitions and notations

Definition

9. A point x in a simple triod X is called the top-point or, briefly, the top of the simple triod X, if there is a homeomorphism $\varphi : ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1]) \to X$ such that $\varphi(0,0) = x$.
10. Let X be a dendroid. A point $x \in X$ is called a ramification-point of the dendroid X, if there is a simple triod T in X with the top x. The set of all ramification-points of X will be denoted by $R(X)$.
11. The continuum X is a fan, if it is a dendroid with at most one ramification point v, which is called the top of the fan X (if it exists).
12. Let X be a fan. For all points x and y in X, we define $A[x,y]$ to be the arc in X with end-points x and y, if $x \neq y$. If $x = y$, then we define $A[x,y] = \{x\}$.
13. Let X be a fan with the top v. We say that that the fan X is smooth if for any $x \in X$ and for any sequence (x_n) of points in X,
$$\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} A[v,x_n] = A[v,x].$$
14. Let X be a fan. We say that X is a Cantor fan, if X is homeomorphic to the continuum $\bigcup_{c\in C} A_c$, where $C \subseteq [0,1]$ is the standard Cantor set and for each $c \in C$, A_c is the convex segment in the plane from $(0,0)$ to $(c,1)$.

Definition

Let (X, f) be a dynamical system. We say that (X, f) is

1. *transitive*, if for all non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.
2. *dense orbit transitive*, if there is a point $x \in X$ such that its trajectory $\{x, f(x), f^2(x), f^3(x), \ldots\}$ is dense in X. We call such a point x *a transitive point in* (X, f).

We say that the mapping f is *transitive*, if (X, f) is transitive.

Definition

Let X be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on X. If F is closed in $X \times X$, then we say that F is *a closed relation on* X.

Definition

Let X be a non-empty compact metric space and let F be a closed relation on X. We call

$$X_F^+ = \left\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \mid \text{ for each } \text{ positive integer } i, (x_i, x_{i+1}) \in F\right\}$$

the Mahavier product of F, and we call

$$X_F = \left\{(\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots) \in \prod_{i=-\infty}^{\infty} X \mid \text{ for each integer } i, (x_i, x_{i+1}) \in F\right\}$$

the two-sided Mahavier product of F.

Basic definitions and notations

Definition

Let X be a non-empty compact metric space and let F be a closed relation on X. The function $\sigma_F^+ : X_F^+ \to X_F^+$, defined by

$$\sigma_F^+(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, x_4, \ldots) \in X_F^+$, is called *the shift map on X_F^+*. The function $\sigma_F : X_F \to X_F$, defined by

$$\sigma_F(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0; x_1, x_2, x_3, \ldots) = (\ldots, x_{-2}, x_{-1}, x_0, x_1; x_2, x_3, x_4, \ldots)$$

for each $(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0; x_1, x_2, x_3, \ldots) \in X_F$, is called *the shift map on X_F*.

Note that σ_F is always a homeomorphism while σ_F^+ may not be a homeomorphism.

Definition

Let X be a compact metric space and let F be a closed relation on X. The dynamical system

1. (X_F^+, σ_F^+) is called *a Mahavier dynamical system*.
2. (X_F, σ_F) is called *a two-sided Mahavier dynamical system*.

We use $\mathbb{X}$ to denote the set

$$\mathbb{X} = ([0,1] \cup [2,3] \cup [4,5] \cup [6,7] \cup \dots) \cup \{\infty\}.$$

We equip $\mathbb{X}$ with the Alexandroff one-point compactification topology $\mathcal{T}$.

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For each non-negative integer k, let $q_k = 1 - \frac{1}{2^k}$ and let

$$X = [q_0, q_1] \cup [q_2, q_3] \cup [q_4, q_5] \cup [q_6, q_7] \cup \ldots \{1\}$$

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(we equip X with the usual topology). Note that the compacta $\mathbb{X}$ and X are homeomorphic.
Let $h : X \to \mathbb{X}$ be any homeomorphism such that for each non-negative integer k, $h(q_k) = k$. On the space $\mathbb{X}$, we always use the metric $\mathrm{d}_{\mathbb{X}}$ that is for all $x, y \in \mathbb{X}$ defined by

$$\mathrm{d}_{\mathbb{X}}(x,y) = |h^{-1}(y) - h^{-1}(x)|.$$

Definition

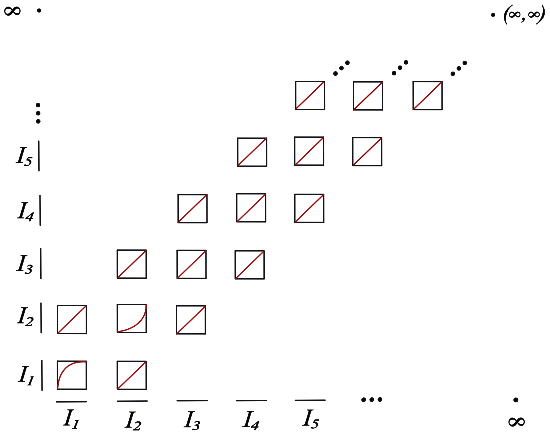
We use the product metric $\mathrm{D}_{\mathbb{X}}$ on the product $\prod_{k=-\infty}^{\infty} \mathbb{X}$, which is defined by

$$\mathrm{D}_{\mathbb{X}}(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\mathrm{d}_{\mathbb{X}}(\mathbf{x}(k), \mathbf{y}(k))}{2^{|k|}} \mid k \text{ is an integer} \right\}$$

for all $\mathbf{x}, \mathbf{y} \in \prod_{k=-\infty}^{\infty} \mathbb{X}$.

We use H to denote the closed relation on $\mathbb{X}$ that is defined as follows:

$$\begin{aligned} H = & \left\{ \left(t, t^{\frac{1}{3}}\right) \mid t \in I_1 \right\} \cup \left\{ \left(t, (t-2)^2 + 2\right) \mid t \in I_2 \right\} \cup \\ & \left\{ (t, t+2) \mid t \in I_1 \cup I_2 \cup I_3 \cup \dots \right\} \cup \left\{ (t, t-2) \mid t \in I_2 \cup I_3 \cup I_4 \cup \dots \right\} \cup \\ & \left\{ (t, t) \mid t \in I_3 \cup I_4 \cup I_5 \cup \dots \right\} \cup \left\{ (\infty, \infty) \right\}. \end{aligned}$$



Theorem

Let X be a compact metric space and let F be a closed relation on X such that $p_1(F) = p_2(F) = X$. The following statements are equivalent.

1. *The map σ_F^+ is transitive.*
2. *The homeomorphism σ_F is transitive.*

Theorem

The dynamical system $(\mathbb{X}_H, \sigma_H)$ is transitive.

A model for $\mathbb{X}_H$

Definition

Let C be the standard middle-third Cantor set in $[0,1]$. For each positive integer k, we use C_k to denote $C_k = C \cap [c_k, d_k]$, where $c_1 = 0$, $d_1 = \frac{1}{3}$, and for each positive integer k, $c_{k+1} = d_k + \frac{1}{3^k}$ and $d_{k+1} = c_{k+1} + \frac{1}{3^{k+1}}$.

For each positive integer k, C_k is a Cantor set and $C = \left(\bigcup_{k=1}^{\infty} C_k \right) \cup \{1\}$. Also, note that for all positive integers k and ℓ, $k \neq \ell \implies C_k \cap C_\ell = \emptyset$.

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In the following theorem, we obtain a model for our two-sided Mahavier product $\mathbb{X}_H$.

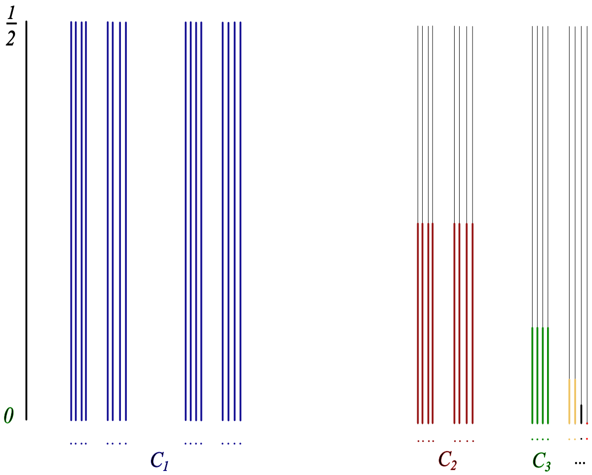
Theorem

There is a homeomorphism

$$\varphi : \mathbb{X}_H \to \left(\bigcup_{k=1}^{\infty}\left(C_k \times \left[0, \frac{1}{2^{2k-1}}\right]\right)\right) \cup \{(1,0)\}$$

such that for each $\mathbf{x} \in \mathbb{X}_H$*, if all the coordinates of* $\mathbf{x}$ *are even, then* $\varphi(\mathbf{x}) = (c, 0)$ *for some* $c \in C$*.*

A model for $\mathbb{X}_H$



Slika: The space $\left(\bigcup_{k=1}^{\infty}\left(C_k \times \left[0, \frac{1}{2^{2k-1}}\right]\right)\right) \cup \{(1,0)\}$

Equivalence relations on $\mathbb{X}_H$

We use $\mathbb{A}$ to denote the uncountable product

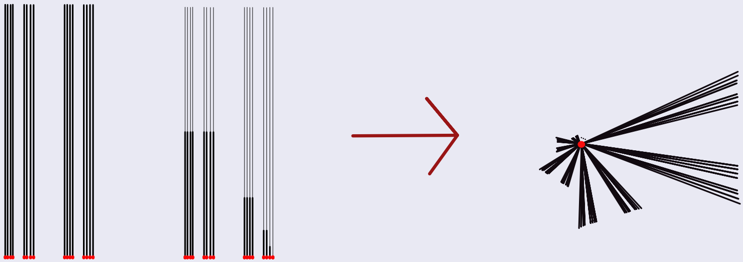
$$\mathbb{A} = \{1,2\} \times \{3,4\} \times \{5,6\} \times \{7,8\} \times \{9,10\} \times \dots$$

Using the set $\mathbb{A}$, we define three relations on $\mathbb{X}_H$.

Definition

We define the relation $\approx$ on $\mathbb{X}_H$ as follows: for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$, we define $\mathbf{x} \approx \mathbf{y}$ if and only if one of the following holds:

1. $\mathbf{x} = \mathbf{y}$,
2. $p_2(\varphi_0(\mathbf{x})) = 0$ and $\varphi_0(\mathbf{y}) = (1,0)$, or $\varphi_0(\mathbf{x}) = (1,0)$ and $p_2(\varphi_0(\mathbf{y})) = 0$,
3. $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y})) = 0$;



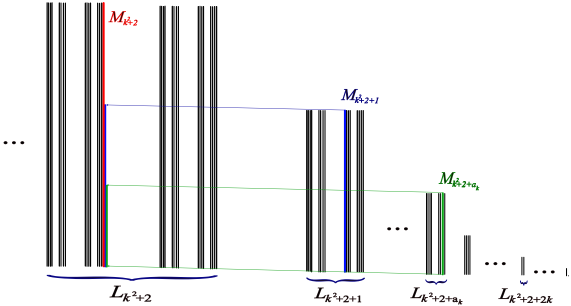
Definition

Let $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathbb{A}$. Then we define the relation $\approx_{\mathbf{a}}$ on $\mathbb{X}_H$ as follows: for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$, we define $\mathbf{x} \approx_{\mathbf{a}} \mathbf{y}$ if and only if one of the following holds:

1. $\mathbf{x} = \mathbf{y}$,
2. there is a positive integer k and there is an $i \in \{1, 2, 3, \ldots, a_k\}$ such that either
   1. $\mathbf{x} \in M_{k^2+2}$ and $\mathbf{y} \in M_{k^2+2+i}$, and
   2. $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y}))$

   or

   1. $\mathbf{y} \in M_{k^2+2}$ and $\mathbf{x} \in M_{k^2+2+i}$, and
   2. $p_2(\varphi_0(\mathbf{x})) = p_2(\varphi_0(\mathbf{y}))$;



Equivalence relations on $\mathbb{X}_H$

Definition

For each $\mathbf{a} = (a_1, a_2, a_3, \dots) \in \mathbb{A}$, we define the relation $\sim_{\mathbf{a}}$ on $\mathbb{X}_H$ by

$$\mathbf{x} \sim_{\mathbf{a}} \mathbf{y} \iff \mathbf{x} \approx \mathbf{y} \text{ or there is } \mathbf{a} \in \mathbb{A} \text{ such that } \mathbf{x} \approx_{\mathbf{a}} \mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$.

Note that $\sim_{\mathbf{a}}$ is an equivalence relation on $\mathbb{X}_H$.

Definition

For each $\mathbf{a} \in \mathbb{A}$, we use $F_{\mathbf{a}}$ to denote the quotient space

$$F_{\mathbf{a}} = \mathbb{X}_H/_{\sim_{\mathbf{a}}}.$$

Theorem

For each $\mathbf{a} \in \mathbb{A}$, $F_{\mathbf{a}}$ is a smooth fan.

Observation

Let $\mathbf{a} \in \mathbb{A}$*. For all* $\mathbf{x}, \mathbf{y} \in \mathbb{X}_H$*,*

$$\mathbf{x} \sim_{\mathbf{a}} \mathbf{y} \quad \Longleftrightarrow \quad \sigma_H(\mathbf{x}) \sim_{\mathbf{a}} \sigma_H(\mathbf{y}).$$

Theorem

Let $\mathbf{a} \in \mathbb{A}$*. The mapping* $\sigma_H^\star : F_{\mathbf{a}} \to F_{\mathbf{a}}$*, defined by*

$$\sigma_H^\star([\mathbf{x}]) = [\sigma_H(\mathbf{x})]$$

for each $\mathbf{x} \in \mathbb{X}_H$*, is a transitive homeomorphism.*

Observation

Note that for each positive integer k*, this transitive homeomorphism* $\sigma_H^\star$*, restricted to* $M_k/_{\sim_{\mathbf{a}}} = \{[\mathbf{x}] \mid \mathbf{x} \in M_k\}$*, is just the identity.*

The family $\mathcal{F}$

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Each member of $\mathcal{F}$ is a smooth fan that admits a transitive homeomorphism. Recall that $\mathbb{A}$ is uncountable.

The family $\mathcal{F}$

Definition

We use $\mathcal{F}$ to denote the family

$$\mathcal{F} = \{F_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{A}\}.$$

Each member of $\mathcal{F}$ is a smooth fan that admits a transitive homeomorphism.
Recall that $\mathbb{A}$ is uncountable.
So, if we show that for all $\mathbf{a}, \mathbf{b} \in \mathbb{A}$,

$$\mathbf{a} \neq \mathbf{b} \quad \Longrightarrow \quad F_{\mathbf{a}} \text{ and } F_{\mathbf{b}} \text{ are not homeomorphic},$$

then this proves that $\mathcal{F}$ is a family of uncountably many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms.

Definition

Let X be a fan with the top o. We define the set $\mathrm{JuMa}(X)$ as follows:

$$\mathrm{JuMa}(X) = \{x \in X \setminus \{o\} \mid \text{ there is a sequence } (e_n) \text{ in } E(X) \text{ such that } \lim_{n\to\infty} e_n = x\}.$$

Definition

Let X be a fan with the top o. For each $e \in E(X)$, we use $A_X[o,e]$ to denote the arc in X from o to e.

JuMa

Proposition

Let X and Y be fans with tops o_X and o_Y, respectively, and let $f : X \to Y$ be a homeomorphism. Then for each $e \in E(X)$,

$$|A_X[o_X, e] \cap \mathrm{JuMa}(X)| = |A_Y[o_Y, f(e)] \cap \mathrm{JuMa}(Y)|.$$

Here $|S|$ denotes the cardinality of S for any set S.

Corollary

Let X and Y be fans with tops o_X and o_Y, respectively. If there is $e \in E(X)$ such that for each $e' \in E(Y)$,

$$|A_Y[o_Y, e'] \cap \mathrm{JuMa}(Y)| \neq |A_X[o_X, e] \cap \mathrm{JuMa}(X)|,$$

then X and Y are not homeomorphic.

Main theorem

Theorem

For all $\mathbf{a}, \mathbf{b} \in \mathbb{A}$,

$$\mathbf{a} \neq \mathbf{b} \quad \Longrightarrow \quad F_{\mathbf{a}} \text{ and } F_{\mathbf{b}} \text{ are not homeomorphic.}$$

Theorem

There is a family of uncountable many pairwise non-homeomorphic smooth fans that admit transitive homeomorphisms.

Thank you!