Localic uniform completions via Cauchy sequences

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Frame presentations

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An advantage frames have over spaces is that, since frames are algebraic structures, they can be presented by generators and relations.

For example, the frame of reals may be presented as

$$\begin{aligned}\mathcal{O}\mathbb{R} = \langle \ell_q, u_q,\ q \in \mathbb{Q} \mid \ell_p = \bigvee_{q>p} \ell_q,\ u_q = \bigvee_{p<q} u_p,\\ \bigvee_{q\in\mathbb{Q}} \ell_q = 1,\ \bigvee_{q\in\mathbb{Q}} u_q = 1,\\ \ell_p \wedge u_q = 0 \ \text{ for } p \geq q,\\ \ell_p \vee u_q = 1 \ \text{ for } p < q\rangle.\end{aligned}$$

Uniform spaces

A uniform space is a set X equipped with a filter $\mathcal{E}$ of binary relations on X satisfying, for all $E \in \mathcal{E}$:

1. $\Delta_X \subseteq E$,
2. $E^o \in \mathcal{E}$,
3. $\exists F \in \mathcal{E}.\ F \circ F \subseteq E$.

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Uniform spaces are a general setting in which to discuss uniform continuity and completeness.

A metric $d \colon X \times X \to \mathbb{R}$ induces a uniformity with basic entourages $\{(x, y) \mid d(x, y) < \varepsilon\}$.

Uniform locales

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Define $v \triangleleft^E u$ (for $u, v \in \mathcal{O}X$, $E \in \mathcal{E}$) to mean $E \circ (v \oplus v) \leq u \oplus u$ and $v \triangleleft^{\mathcal{E}} u$ to mean $v \triangleleft^E u$ for some $E \in \mathcal{E}$.

A pre-uniform locale $(X, \mathcal{E})$ is a uniform locale if $u = \bigvee_{v \triangleleft^{\mathcal{E}} u} v$ for all u.

Completeness

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The completion of a uniform space/locale is usually constructed in terms of (regular) Cauchy filters.

A regular Cauchy filter on a uniform locale $(X, \mathcal{E})$ is a proper filter F on $\mathcal{O}X$ such that

- for every $E \in \mathcal{E}$, there is some $u \in F$ with $u \oplus u \leq E$,
- if $u \in F$ then there is a $v \in F$ such that $v \triangleleft^{\mathcal{E}} u$.

The locale of regular Cauchy filters

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There is an obvious locale embedding $\gamma \colon X \hookrightarrow \mathcal{C}X$ obtained by sending $[u \in F] \in \mathcal{O}\mathcal{C}X$ to $u \in \mathcal{O}X$. This is the completion of X.

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Metric spaces have uniform structures and the uniform completion agrees with the familiar metric completion.

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But Cauchy sequences do work in the pointfree setting!

Modulated Cauchy sequences

A Cauchy sequence in a uniform space $(X, \mathcal{E})$ is a map $s\colon \mathbb{N} \to X$ such that $\forall E \in \mathcal{B}.\ \exists N \in \mathbb{N}.\ \forall n, n' \geq N.\ (s(n), s(n')) \in E$ where $\mathcal{B}$ is some chosen base for the uniformity $\mathcal{E}$.

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We can now define a locale of modulated Cauchy sequences.

The locale of modulated Cauchy sequences

Let $X = (X, \mathcal{E})$ be a uniform locale with base $\mathcal{B} \subseteq \mathcal{E}$. We give a presentation for the frame of $\mathrm{ModCauchy}(X)$.

The generators are:

- $[s(n) \in u]$ for each $n \in \mathbb{N}$ and $u \in \mathcal{O}X$,
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The relations are:

- $\bigvee_\alpha \bigwedge_{u \in F_\alpha} [s(n) \in u] = [s(n) \in \bigvee_\alpha \bigwedge F_\alpha]$ for each family $(F_\alpha)_\alpha$ of finite subsets of $\mathcal{O}X$,
- $1 \leq \bigvee_{k \in \mathbb{N}} [m(E) = k]$ for each $E \in \mathcal{B}$,
- $[m(E) = k] \leq \bigvee_{u \oplus u' \leq E} [s(n) \in u] \wedge [s(n') \in u']$ for $E \in \mathcal{B}$, $k \in \mathbb{N}$ and $n, n' \geq k$.

The limit map

We now define a map from $\mathrm{ModCauchy}(X)$ to the completion of X, which 'takes the limit' of the Cauchy sequences.

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This map $q\colon \mathrm{ModCauchy}(X) \to \mathcal{C}X$ is given by

$$q^*([u \in F]) = \bigvee_{E \in \mathcal{B}} \bigvee_{v \lhd^E u' \lhd^{\mathcal{E}} u} \bigvee_{k' \leq k \in \mathbb{N}} [m(E) = k'] \wedge [s(k) \in v].$$

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Intuitively, this says $q((s,m))$ lies in u iff $s(k) \in v$ for some $k \in \mathbb{N}$ and $v \lhd^E u' \lhd^{\mathcal{E}} u$ such that $m(E) = k' \leq k$.

The limit map as a quotient

We claim $q\colon \mathrm{ModCauchy}(X) \to \mathcal{C}X$ is a well-behaved quotient map.

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More formally, we define a join-preserving map $g\colon \mathcal{O}\mathrm{ModCauchy}(X) \to \mathcal{O}\mathcal{C}X$ such that $gq^* = \mathrm{id}_{\mathcal{O}\mathcal{C}}$ and $g(a \wedge q^*(b)) = g(a) \wedge b$. This exhibits q as a triquotient map.

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I will omit the definition of g, but intuitively it associates a point p of the completion to a collection of modulated Cauchy sequences that converge to p 'sufficiently quickly'.

The upshot

Note that we can now throw away our old construction of $\mathcal{C}X$ and recover the completion by taking a quotient by the corresponding internal equivalence relation on $\mathrm{ModCauchy}(X)$.

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Moreover, note that $\mathrm{ModCauchy}(X)$ can be highly non-spatial when $\mathcal{B}$ is uncountable. Intuition: $\mathbb{N}^{\mathcal{B}}$ is non-spatial for uncountable $\mathcal{B}$.

References

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2. G. Manuell, *Sequences suffice for pointfree uniform completions*, to appear on the arXiv soon!