Disjoint π-bases on compact spaces

Joint work with Alan Dow
38th Summer Conference in Topology and Its Applications

Hector BA
UNC Charlotte

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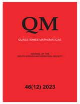
Plan: two questions of Gruenhage and Tkachuk

Q1: Does every compact (Fréchet-Urysohn) space of countable tightness have a countable disjoint local π-base at every point?

Q2: Suppose that X is a (hereditarily) Lindelof space for which the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subseteq X$. Must X be discretely selective?



Quaestiones Mathematicae

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Discrete selectivity, shrinking properties, and disjoint local π-bases

Gary Gruenhage & Vladimir V. Tkachuk

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- The π-**character** of X is $\pi\chi(X) := \sup\{\pi\chi(x, X) : x \in X\}$.
- The **tightness** of X is the least upper bound cardinal $t(X)$ so that for all $A \subseteq X$,

$$\overline{A} = \bigcup\{\overline{B} : B \in [A]^{\leq t(X)}\}$$

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Remark: If a space X has the disjoint shrinking property and X has a countable local π-base at a point $x \in X$, then there exists a disjoint local π-base at the point x.

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Remark: If a space X has the disjoint shrinking property and X has a countable local π-base at a point $x \in X$, then there exists a disjoint local π-base at the point x.

- A space X has the **disjoint (discrete) shrinking property** if for every sequence $\{U_n : n \in \omega\}$ of open sets there are open sets $V_n \subseteq U_n$ so that $\{V_n : n \in \omega\}$ is disjoint (discrete).

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- Under PFA, every compact space of countable tightness has a **countable disjoint local π-base at every point**.
- If X is a compact space of countable tightness. If every open subspace of X is non-separable, then X has a **countable disjoint local π-base at every point**.

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Theorem(Tkachuk, Wilson, 2019): A compact space has a disjoint local π-base at a point $x \in X$ if and only if $X \setminus \{x\}$ is not cellular-compact.

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- ▷ A sequence $\{x_\alpha : \alpha < \kappa\}$ is **free** if $\overline{\{x_\gamma : \gamma < \alpha\}} \cap \overline{\{x_\gamma : \gamma \geq \alpha\}} = \emptyset$ for all $\alpha < \kappa$.

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   - 3.1 Z_β^α is a G_δ-set;
   - 3.2 $Z_\beta^\alpha \supseteq Z_\beta^{\alpha+1}$;
   - 3.3 $Z_\alpha^\alpha = \bigcap\{f^{-1}([0,1)) : f \in F \cap M_\alpha\}$,
   - 3.4 $f_\alpha^{-1}([0,1)) \not\supseteq Z_\beta^\alpha$;
   - 3.5 $Z_\beta^{\alpha+1} = Z_\beta^\alpha \cap f_\alpha^{-1}(\{1\})$, for $\beta < \alpha$, and $Z_\alpha^{\alpha+1} = Z_\alpha^\alpha \cap f_\alpha^{-1}([0,1))$;
   - 3.6 $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, if α is limit; and
   - 3.7 $M_\alpha, f_\alpha \in M_{\alpha+1}$, (hence $F \cap M_\alpha$ and $\{Z_\beta^\alpha : \beta < \alpha\}$ are in $M_{\alpha+1}$).

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Apply elementarity to find $f_\gamma \in M_{\gamma+1}$ with the same property. For every $\beta < \gamma$, let

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Continuing for successor $\gamma + 2$:

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Claim: At any successful stage γ, the family $\{Z_\beta^\gamma : \beta < \gamma\}$ is discrete.

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$\rightarrow$ $A \subseteq X$ has non-empty interior then $\pi\chi(A, X) = 1$.

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→ A σ-compact space X has the discrete shrinking property if and only if $\pi\chi(K, X) > \omega$ for any compact subspace $K \subseteq X$.

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7. $\{y_\alpha : \alpha < \omega_1\}$ (is Lindelof) has open cover $\{W_\alpha : \alpha < \omega_1\}$ with no countable subcover: if $\{W_\xi : \xi < \alpha\}$ was a cover, then y_α is in W_ξ for some $\xi < \alpha$ which is not possible since S_α is almost disjoint from W_ξ.

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There are lots of partial sequences that are closed discrete.
How can we get a full one?