A homological bound on entropy in arbitrary compact spaces

38th Summer Conference on Topology and its Applications

joint work with:

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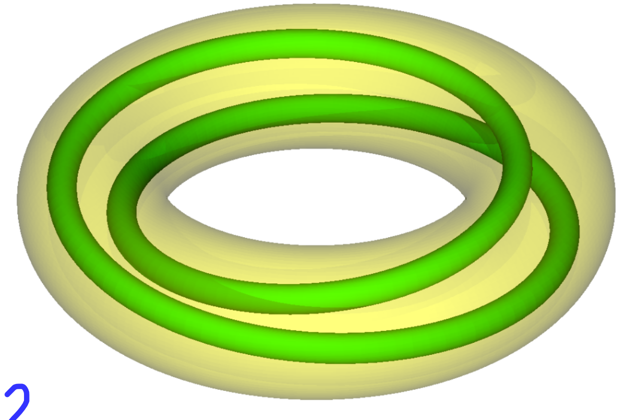
(X,f) dynamical system, X compact, $f: X \to X$ continuous map

Topological entropy $h_{top}(f) \in [0,+\infty]$

- Measures the complexity
- Local and/or global phenomena
- Difficult to compute, enough >0
- Entropy conjecture

$X = \bigcap_{n \geq 0} g^n(T)$ $\quad T \supset g(T)$



Examples:

- $f: S^1 \to S^1$, $z \mapsto z^2$

  $h_{top}(f) = \log 2$

- X = dyadic solenoid

  $g|_X : X \to X$

  $h_{top}(g|_X) = \log 2$

Manning's theorem

X = compact manifold

$$f_*: H_1(X;\mathbb{C}) \to H_1(X;\mathbb{C})$$

λ eigenvalue of f_* then $h_{top}(f) \geq \log|\lambda|$

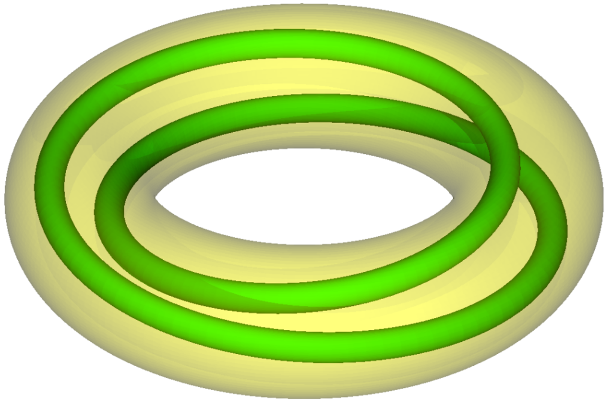
Example: $f: S^1 \to S^1$, $z \mapsto z^2$

$$H_1(S^1;\mathbb{C}) \cong \mathbb{C}$$

$$f_*: \mathbb{C} \to \mathbb{C}, \quad w \mapsto 2\cdot w$$

$$\overset{\text{Manning}}{\Longrightarrow} \quad h_{top}(f) \geq \log 2$$

Other compact sets?



$H_1(X;\mathbb{Z}) = 0$

Extension of Manning?

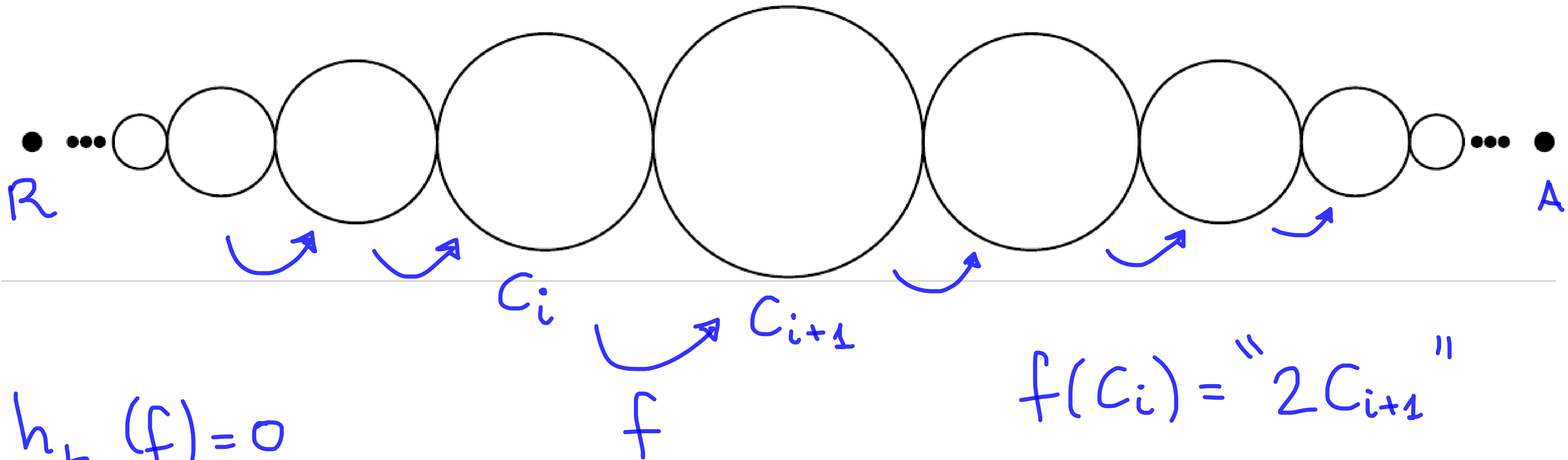
Cech type theories have a word:

$\check{H}_1(X;\mathbb{K}) \cong \mathbb{K}$ $\quad\mathbb{K}$ a field

$g_* : \check{H}_1(X) \to \check{H}_1(X) \Rightarrow h_{top}(g) \geq \log 2$ ✓

$g_* \equiv$ "$\cdot 2$"

Cohomology >> Homology



$$f(C_i) = \text{“} 2\, C_{i+1} \text{”}$$

$$h_{top}(f) = 0$$

but $f_*\left(\sum_{j\in\mathbb{Z}} C_j\right) = 2\cdot\sum_{j\in\mathbb{Z}} C_j \rightsquigarrow 2$ is an eigenvalue of $f_* : \check{H}_1(X) \circlearrowleft$

$f^* : \check{H}^1(X) \to \check{H}^1(X)$ has no eigenvalues

Manning's theorem (classical)

X = compact manifold

$$f_* : H_1(X;\mathbb{C}) \to H_1(X;\mathbb{C})$$

λ eigenvalue of f_* then $h_{top}(f) \geq \log|\lambda|$

Manning's theorem (revisited)

X = compact and locally connected

$$f_* : \check{H}^1(X;\mathbb{C}) \to \check{H}^1(X;\mathbb{C})$$

λ eigenvalue of f_* then $h_{top}(f) \geq \log|\lambda|$

Manning's theorem (classical)

X = compact manifold

$$f_* : H_1(X;\mathbb{C}) \to H_1(X;\mathbb{C})$$

λ eigenvalue of f_* then $h_{top}(f) \geq \log|\lambda|$

Manning's theorem (revisited)

X = compact

$$f_* : \check{H}^1(X;\mathbb{C}) \to \check{H}^1(X;\mathbb{C})$$

λ eigenvalue of f_* then $h_{top}(f) \geq \dfrac{\log|\lambda|}{d}$

where d = degree of λ as algebraic number

Idea of Manning's proof:

Bound length of paths $\{\gamma_n \sim f^n(\gamma) : n \in \mathbb{N}\}$

$\mathrm{length}(\gamma_n) \lessapprox S(n, \varepsilon)$

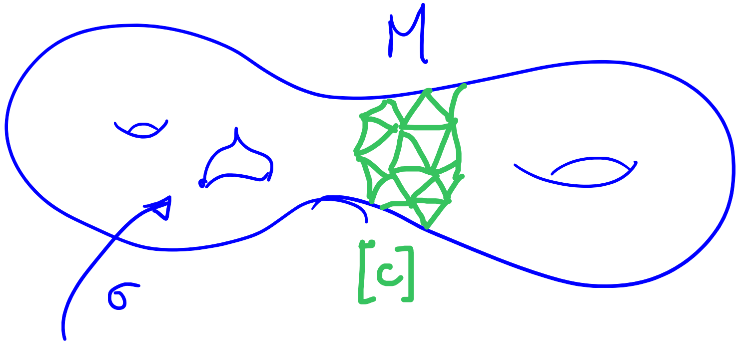
Our setting:

How do we measure "length" of a (Cech) cocycle?
Or of a (Cech) cycle?

Is there a natural pairing $\check{H}^*(X;\mathbb{C}) \times \check{H}_*(X;\mathbb{C}) \longrightarrow \mathbb{C}$?

More dramatic:
how to describe $\boxed{\text{intrinsically}}$ a (Cech) 1-cocycle?
1-cycle?

Integration in singular homology/cohomology



Δ^q forms can be integrated along (smooth) simplices

Singular cohomology $H^*_{dR}(M)$

Cocycles = closed differential forms

Singular homology

Cycles = submanifolds

$$H^q_{dR}(M) \times H_q(M) \longrightarrow \mathbb{K}$$
$$([\omega], [c]) \longmapsto \int_c \omega$$

De Rham: bilinear non-degenerate pairing

Goal: pair Cech homology/cohomology via integration

Cech cohomology after Alexander-Spanier

1-cochains: $\boxed{\begin{aligned} \phi : X \times X &\longrightarrow \mathbb{C} \\ (x_0, x_1) &\longmapsto \phi(x_0, x_1) \end{aligned}}$ + only care behavior close to diagonal

$$\delta\phi : X \times X \times X \longrightarrow \mathbb{C}$$

$$\delta\phi(x_0, x_1, x_2) = \phi(x_1, x_2) - \phi(x_0, x_2) + \phi(x_0, x_1)$$

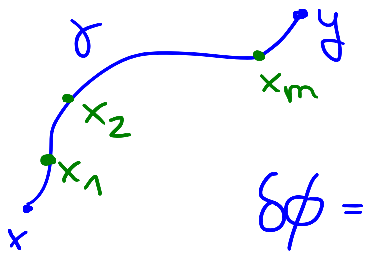
1-cocycles: $\delta\phi = 0$ if x_0, x_1, x_2 are close

$$\check{H}^1(X; \mathbb{C}) = \text{1-cocycles} \big/ \text{1-coboundaries}$$

Integration of 1-cocycles = Riemann integration

$\phi : X \times X \longrightarrow \mathbb{K}$

(repr. of) 1-cocycle



$\delta\phi = 0$ locally

$$0 = \delta\phi(x, x_1, x_2) = \phi(x_1, x_2) - \phi(x, x_2) + \phi(x, x_1)$$

if x, x_1, x_2 are close

Integral along γ :

$$\int_\gamma \phi := \underbrace{\phi(x, x_1) + \phi(x_1, x_2)} + \dots + \phi(x_m, y) =$$

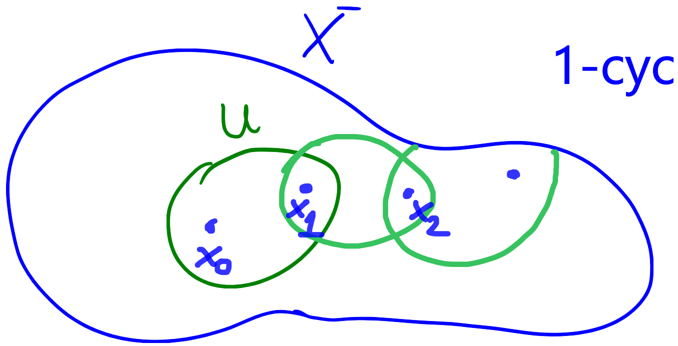
$$= \phi(x, x_2) + \dots + \phi(x_m, y) = \dots$$

Independent of the partition if small enough

Cech 1-homology after Vietoris

$\mathcal{U}$ fixed open cover of X

scale below which topology is disregarded



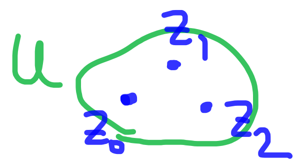
1-cycle = comb. of $(x_0\, x_1) + (x_1\, x_2) + \ldots + (x_k\, x_0)$

s.t. $(x_i\, x_{i+1}) \in U_i \in \mathcal{U}$

$\partial(x_i\, x_{i+1}) = x_{i+1} - x_i$

$$H_1^{\mathcal{U}}(X) = \frac{\mathcal{U}\text{-small 1-cycles}}{\mathcal{U}\text{-small 1-boundaries}}$$

$\partial(z_0 z_1 z_2) = (z_1 z_2) - (z_0 z_2) + (z_0 z_1)$



$$\varprojlim_{\mathcal{V}} H_1^{\mathcal{V}}(X) \cong \check{H}_1(X)$$

refinement of cycles (finer partition)

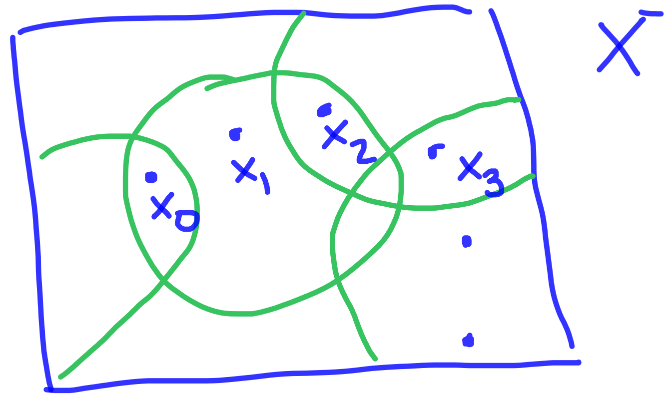
Integration of 1-cocycles over 1-cycles

1-cocycle $\phi : X \times X \to \mathbb{C}$

$\delta\phi = 0$ locally $\Rightarrow$ $\exists\, \mathcal{U}$ open cover of X s.t.

$\delta\phi(x_0 x_1 x_2) = 0 \quad \forall\, x_0 x_1 x_2 \ \mathcal{U}\text{-close}$

$$\check{H}_1(X) = \lim_{\substack{\longleftarrow \\ \mathcal{V} \text{ open cover}}} H_1^{\mathcal{V}}(X)$$



($\mathcal{U}$-small) 1-cycle

$c_{\mathcal{U}} = (x_0 x_1) + (x_1 x_2) + \ldots + (x_k x_0)$ that represents $\gamma \in \check{H}_1(X)$ at scale $\mathcal{U}$

$$\int_\gamma \phi := \sum_{i=0}^{k} \phi(x_i x_{i+1})$$

Integration as a pairing

Theorem

$$\check{H}^1(X;\mathbb{C}) \times \check{H}_1(X;\mathbb{C}) \longrightarrow \mathbb{C}$$

$$[\phi] \quad , \quad \gamma = [c_u] \longmapsto \int_\gamma \phi$$

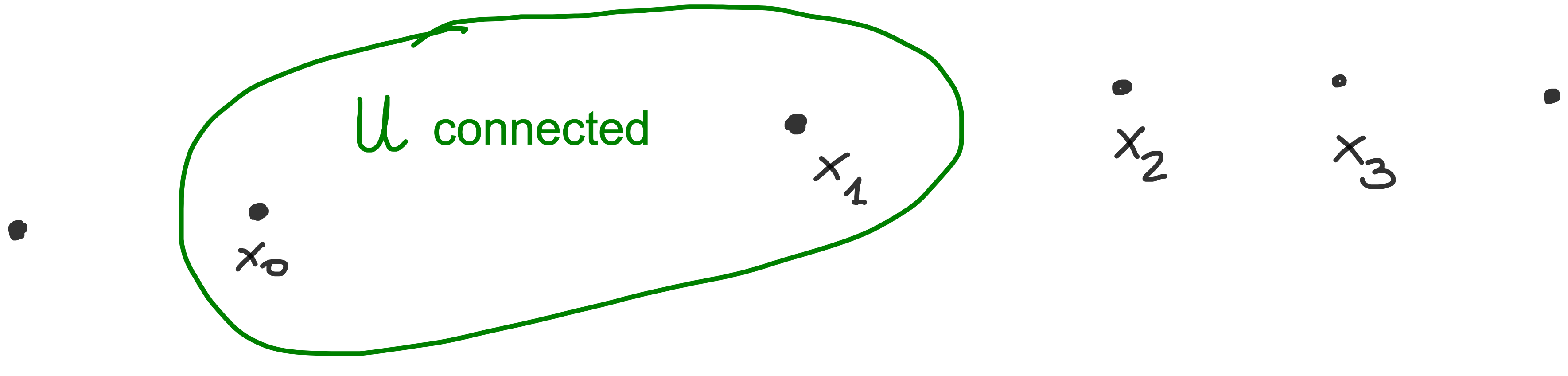
is a well-defined non-degenerate bilinear pairing

The pairing works for any dimension:

$\check{H}_*(X;\mathbb{C})$ is dual to $\check{H}^*(X,\mathbb{C})$

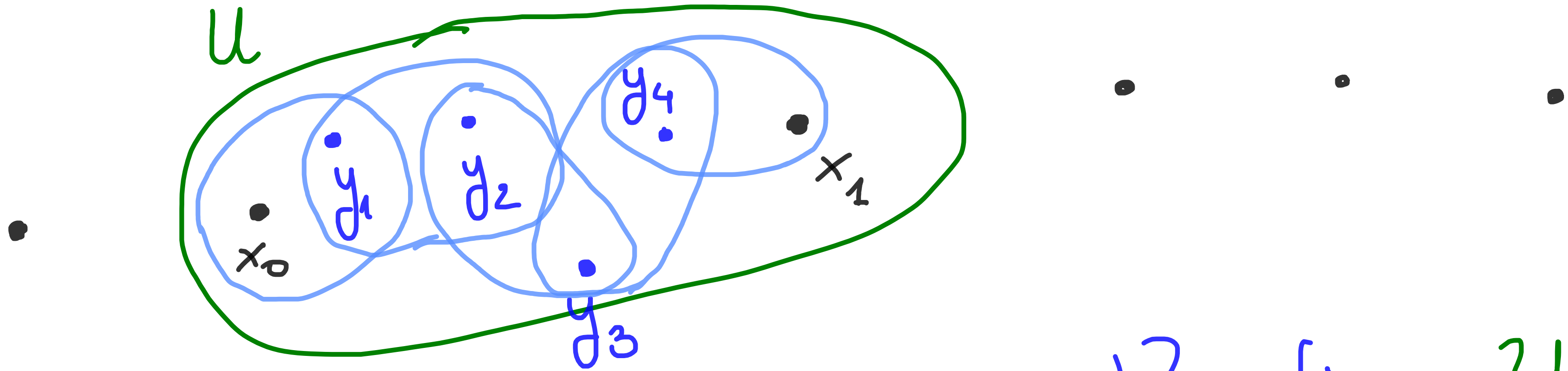
and conversely (if finite dim.)

Refinement of 1-cycles



$$(x_0 x_1) + (x_1 x_2) + \cdots$$

Refinement of 1-cycles



$\mathcal{V}$ refines $\mathcal{U}$

at scale $\mathcal{U}$

$$(x_0 x_1) = (x_0 y_1) + (y_1 y_2) + (y_2 y_3) + \dots + (y_k x_1)$$

$\mathcal{U}$-small

$\mathcal{V}$-small $< \# \mathcal{U}$ pieces

Sketch of proof of Manning (revisited) - case locally conn.

- λ eigenvalue of $f^*\colon \check{H}^1(X,\mathbb{C}) \to \check{H}^1(X,\mathbb{C})$
- ϕ 1-cocycle s.t. $f^*(\phi) = \lambda\phi$

  $\delta\phi = 0 \rightsquigarrow$ open cover $\mathcal{U}$ (finite & connected sets)
- c 1-cycle $\mathcal{U}$-small s.t. $\int_c \phi \neq 0$
- Fix $n \in \mathbb{N}$, $\mathcal{U}_n = \bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}$ $\quad c \underset{\text{refines}}{\rightsquigarrow} c_n$ is $\mathcal{U}_n$-small

  $\leq \#\mathcal{U}_n$ pieces

$$\left|\lambda^n \int_c \phi\right| = \left|\lambda^n \int_{c_n} \phi\right| = \left|\int_{c_n} f^{*n}\phi\right| = \left|\int_{f_*^n c_n} \phi\right| \leq \max \phi \cdot \#\mathcal{U}_n$$

$\mathcal{U}$-small

$$\Downarrow$$

$\#\mathcal{U}_n$ grows $\geq |\lambda|^n$

Muito obrigado!