Cartesian closed stable subcategories of $[0, 1]$-Cat

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Outline

$[0,1]$-Cat as an extension of Ord

Cartesian closed stable subcategories of $[0,1]$-Cat

Quantales and quantale-valued categories

$(Q, \&, k)$ A commutative and unital quantale
(a commutative monoid in Sup)

$p\&q \leq r \iff q \leq p \to r$

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A functor $f : (A, r) \longrightarrow (B, s)$ is a map $f : A \longrightarrow B$ such that

$$r(x, y) \leq s(f(x), f(y))$$

for all x and y in A. The category consisting of all Q-categories and functors is written as

Q-Cat.

Ordered sets and metric spaces as Q-ordered sets

Let $(Q, \&, k) = (\{0, 1\}, \wedge, 1)$.
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preordered set:

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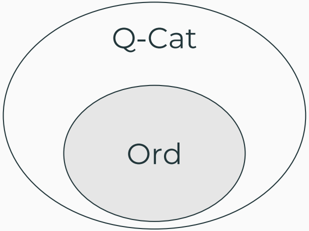
Let $(Q, \&, k) = ([0,\infty]^{\mathrm{op}}, +, 0)$.
A Q-category (X, d) is exactly a generalized metric space.

- $d(x,x) = 0$,
- $d(x,y) + d(y,z) \geq d(x,z)$.

Ord **is a stable subcategory of** Q-Cat

A given preordered set $(X, \leq)$ can be treated as a Q-category naturally. Moreover, the category Ord can be embedded in Q-Cat as a full subcategory.



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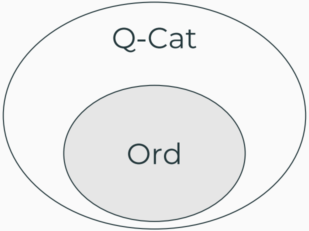


$$\begin{tikzcd} \mathsf{Ord} \arrow[r, tail, shift left=0] & \textsf{Q-Cat} \arrow[l, two heads, shift right=3, "L"'] \arrow[l, two heads, shift left=3, "R"] \end{tikzcd}$$

If $(Q, \&, k)$ is integral, that is, $k = \top$, then the embedding functor π : Ord $\longrightarrow$ Q-Cat has both a concrete left adjoint L and a concrete right adjoint R.

Real-enriched category

Let $(Q, \&, k) = ([0,1], \&, 1)$, and $\&$ be a continuous function. In this case, a Q-category is often called a real-enriched category, and it is written as a $[0,1]$-category.

The category consisting of all real-enriched categories is denoted by

$$[0,1]\text{-Cat},$$

which is Cartesian closed if and only if

$$\& = \wedge.$$

When $\&$ is the Łukasiewicz t-norm

Let

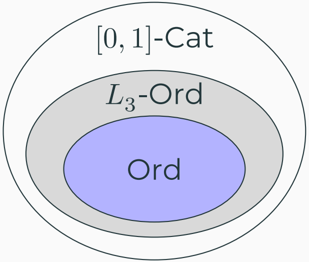
$$x \odot y = \max\{x + y - 1, 0\},$$

then $\odot$ is the Łukasiewicz t-norm. Let

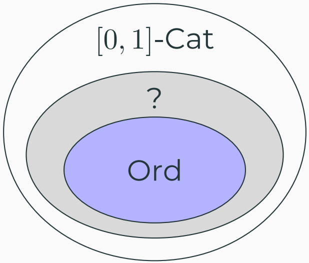
$$L_3 = \{0, 0.5, 1\},$$

Then $(L_3, \odot, 1)$ is a subquantale of $([0,1], \odot, 1)$. L_3-Cat is Cartesian closed, and it is a stable subcategory of $[0,1]$-Cat.

$$\& = \odot$$



A question



Question

For a continuous t-norm $\&$ on $[0,1]$, is there any Cartesian closed stable subcategory of $[0,1]$-Cat containing Ord?

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Stable subcategories of $[0,1]$-Cat

Recall that, if $\mathcal{A}$ is a stable subcategory of Q-Cat, then the inclusion map $i : \mathcal{A} \rightarrowtail$ Q-Cat has both a concrete left adjoint L and a concrete right adjoint R.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, tail] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, phantom, shift right=3ex, "\perp"] & \text{Q-Cat} \arrow[l, two heads, shift right=6ex, "L"'] \arrow[l, two heads, shift left=6ex, "R"]
\end{tikzcd}
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Stable subcategories of $[0,1]$-Cat

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$$\begin{tikzcd} \mathcal{A} \arrow[r, rightarrowtail] & \text{Q-Cat} \arrow[l, two heads, bend right, "L"'] \arrow[l, two heads, bend left, "R"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_X \arrow[r, rightarrowtail] & \text{Q-Cat}_X \arrow[l, two heads, bend right, "L_X"'] \arrow[l, two heads, bend left, "R_X"] \end{tikzcd}$$

The fibre $\mathcal{A}_X$ of a set X is consisting of all $\mathcal{A}$-objects with the underlying set X. For each set X, the fibre $\mathcal{A}_X$ is a complet lattice equipped with the order

$$(X, r) \leq (X, s) \iff \forall x, y \in X, r(x, y) \leq s(x, y),$$

since Q-Cat is fibre complete, that is, each Q-Cat$_X$ is a complete laittice.

The fibre $\mathscr{A}_{\mathbf{2}}$

Let $\mathbf{2}$ be the set $\{0, 1\}$. Then the fibre $[0, 1]$-Cat$_{\mathbf{2}}$ is consisting of all $[0, 1]$-orders on the set $\mathbf{2}$. Clearly, $[0, 1]$-Cat$_{\mathbf{2}} \cong [0, 1]^2$.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd}$$

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Therefore, $\mathcal{A}_{\mathbf{2}}$ can be regarded as a subset of $[0, 1]^2$, which satisfies that

1. $(a_i, b_i) \in \mathcal{A}_{\mathbf{2}} \Longrightarrow \sup(a_i, b_i) \in \mathcal{A}_{\mathbf{2}}, \inf(a_i, b_i) \in \mathcal{A}_{\mathbf{2}}$;
2. $(a, b) \in \mathcal{A}_{\mathbf{2}} \Longrightarrow (b, a) \in \mathcal{A}_{\mathbf{2}}$;
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Moreover, a stable full subcategory $\mathcal{A}$ of $[0,1]$-Cat is determined completely by a subset $\Gamma \subseteq [0,1]^2$, which satisfies the above three conditions.

$$(X, r) \in \mathcal{A} \iff \forall x, y \in X, (r(x,y), r(y,x)) \in \Gamma.$$

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- R. Lowen, *Index analysis, approach theory at work*, Springer, 2015.

The main result

The set

$$M = \{x \in [0,1] \mid x \& x = x\}$$

is a closed subset of $[0,1]$ containing both 0 and 1. Furthermore, one has that $x \& y \in M$ if $x, y \in M$. Hence, $(M, \&, 1)$ is a subquantale of $([0,1], \&, 1)$.

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Theorem
Let $\&$ be a continuous t-norm on $[0,1]$. A stable subcategory of $[0,1]$-Cat *is Cartesian closed if and only if*

$$\mathcal{A}_{\mathbf{2}} \subseteq M^2.$$

Therefore, the largest $\mathcal{A} =$ M-Cat.

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- $\& = \odot$, $M = [0, 0.5] \cup \{1\}$.
- $\& = \cdot$, the multiplication. $M = \{0, 1\}$ and M-Cat $=$ Ord.
  Since $([0,1], \cdot, 1) \cong ([0,\infty]^{\mathrm{op}}, +, 0)$, the largest Cartesian closed stable subcategory in $\mathbf{Met}$ is Ord.

Thank You!