On w-unicoherence, top-irreducibility and n-cells at the top

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joint work with
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We say that a continuum X is *weakly unicoherent*, or simply *w-unicoherent*, provided that there exists $\varepsilon > 0$ such that for each proper subcontinua A and B of X with $A, B \in B_{H_d}(X, \varepsilon)$ and $X = A \cup B$, it is satisfied that $A \cap B$ is connected.

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Let X be a continuum. Then the following statements are equivalent.

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There exists a continuum X that is w-unicoherent, it is not top-irreducible and such that for any decomposition of $X = X_1 \cup X_2$ satisfying that $X_1 \cap X_2$ is connected, the intersection is not terminal in either X_1 or X_2.

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Given a positive integer n, characterize continua X for which X has a neighborhood in $C(X)$ that is an n-cell.

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Let X be a continuum, let μ be a Whitney map for $C(X)$ and $t_0 \in (0,1)$. If $\mu^{-1}(t_0)$ is a tree having n end points, then the following statements hold.

1. $\mu^{-1}(t)$ is homeomorphic to some positive Whitney level of $\mu^{-1}(t_0)$ for each $t \in (t_0, 1)$.
2. There is $t_1 > t_0$ such that
    1. $\mu^{-1}([t_1, 1])$ is an n-cell;
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    2. $\mu^{-1}(t)$ is a $(n-1)$-cell for each $t \in (t_1, 1)$.

n-cells at the top

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Let X be a continuum that can be expressed as $X = \bigcup_{i=0}^{n} P_i$, where $P_0, P_1, P_2, \ldots, P_n$ are hereditarily indecomposable continua, $P_i \cap P_j = \varnothing$ and $P_i \cap P_0$ is a proper subcontinuum of P_i contained in a different composant of P_0 than $P_j \cap P_0$ for each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Then X has a positive Whitney level which is a tree.

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Theorem (Ahuatzin-Ordoñez-Villanueva, 2024)

Suppose that $\mu^{-1}(t_0)$ is a positive Whitney level for $C(X)$ that is a simple n-od. Then X is an n-od; moreover, there exist subcontinua X_1, X_2, …, X_n of X and $C \in \mu^{-1}(t_0)$ such that

(a) $X = X_1 \cup \cdots \cup X_n$ and $X_i \cap X_j = C$ for each $i \neq j$;

(b) X_i is decomposable for each $i \in \{1, \ldots, n\}$;

(c) X_i is w-unicoherent for each $i \in \{1, \ldots, n\}$;

(d) X_i is top-irreducible for each $i \in \{1, \ldots, n\}$;

(e) C contains one point of irreducibility of X_i for each $i \in \{1, \ldots, n\}$.

Persistent properties

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Question

If $\mu^{-1}(t_0)$ is a positive Whitney level for $C(X)$ that is an arc, is it true that for any other Whitney map ω for $C(X)$ there exists $s_0 \in (0,1)$ which $\omega^{-1}(s_0)$ is also an arc?

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The answer is "yes".

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Theorem (Ahuatzin-Ordoñez-Villanueva, 2024)

Suppose that $\mu^{-1}(t_0)$ is a positive Whitney level for $C(X)$ that is a simple close curve and let ω another Whitney map for $C(X)$, then there exists $s_0 \in (0,1)$ which $\omega^{-1}(s_0)$ is also a simple closed curve.

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Example – (Ahuatzin-Ordoñez-Villanueva, 2024)

There exists a continuum X, a Whitney map μ for $C(X)$ and a positive Whitney level $\mu^{-1}(t_0)$ that is a tree, for which there exists another Whitney map ω for $C(X)$ such that no Whitney level of ω is a tree.

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Obrigado!