Quantale-enriched lower separation axioms and the principle of enriched continuous extension

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Outline of the talk

1. Background: quantales, the category **Sup**($\mathfrak{Q}$) and module theory over **Sup**.
2. $\mathfrak{Q}$-enriched topological spaces. Examples.
3. Lower separation axioms.
4. (Weak) regularity and the principle of continuous extension.
5. Example: the interval topology on projective $\mathfrak{Q}$-modules in **Sup**.

Background

- **Sup** will denote the category of complete lattices and join-preserving maps.
- **Sup** is a monoidal closed category.
- Semigroups (monoids) in **Sup** are known as (unital) quantales.
- Unital quantales can also be seen as small, complete, thin, skeletal and monoidal closed categories.
- Explicitly, a **unital quantale** $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ is a complete lattice $\mathfrak{Q}$ together with an associative operation $*\colon \mathfrak{Q} \times \mathfrak{Q} \to \mathfrak{Q}$ which preserves joins in each variable separately, and such that e is the unit w.r.t. this operation.
- Unless otherwise stated, $\mathfrak{Q}$ will denote a (not necessarily commutative) unital quantale and e will denote the unit.
- A **quantale homomorphism** is a map which preserves arbitrary joins and the quantale operation. A **strong homomorphism** additionally preserves the top.

An element δ of a quantale $\mathfrak{Q}$ is said to be

- **dualizing** if

$$\delta \swarrow (a \searrow \delta) = a = (\delta \swarrow a) \searrow \delta, \qquad \forall a \in \mathfrak{Q},$$

- **cyclic** if

$$a \searrow \delta = \delta \swarrow a, \qquad \forall a \in \mathfrak{Q}.$$

A quantale is **Girard** if it has a cyclic and dualizing element.

A quantale is **integral** if $e = \top$.

In an integral quantale, any dualizing element must coincide with $\perp$.

Given a quantale $\mathfrak{Q}$, we denote by $\mathfrak{Q}^{op}$ the **opposite quantale** with multiplication

$$x *_{op} y := y * x.$$

Background

A **$\mathfrak{Q}$-enriched category** is a pair (X, p) where X is a set and $p\colon X \times X \to \mathfrak{Q}$ is a map satisfying.

1. $e \leq p(x, x)$ for all $x \in X$,
2. $p(y, z) * p(x, y) \leq p(x, z)$ for all $x, y, z \in X$.

A **$\mathfrak{Q}$-functor** $f\colon (X, p) \to (Y, q)$ is a map $f\colon x \to Y$ such that $p(x, y) \leq q(f(x)), f(y))$ for all $x, y \in X$.

The category of $\mathfrak{Q}$-enriched categories and $\mathfrak{Q}$-functors between them will be denoted by $\mathbf{Cat}(\mathfrak{Q})$.

Furthermore, consider the category $\mathbf{Sup}(\mathfrak{Q})$ of cocomplete, separated $\mathfrak{Q}$-enriched categories and cocontinuous $\mathfrak{Q}$-functors between them.

Recall that a $\mathfrak{Q}$-enriched category is **cocomplete** if and only if the $\mathfrak{Q}$-enriched Yoneda embedding $\mathbf{y}_{(X,p)}\colon (X, p) \to \mathbb{P}(X, p)$ has a left adjoint $\mathfrak{Q}$-functor $\mathbf{Sup}_{(X,p)}\colon \mathbb{P}(X, p) \to (X, p)$. The relation $\mathbf{Sup}_{(X,p)}(f) = \mathbf{Colim}^f(1_X)$ holds.

Background: right $\mathfrak{Q}$-modules

A **right $\mathfrak{Q}$-module** in **Sup** is a complete lattice M provided with a right action $M \otimes \mathfrak{Q} \xrightarrow{\boxdot} M$ in the monoidal closed category **Sup**. Due to the universal property of the tensor product, every right action on M can be identified with a map $M \times \mathfrak{Q} \xrightarrow{\boxdot} M$, which is join-preserving in each variable separately and satisfies the following axioms:

$$m \boxdot e = m \quad \text{and} \quad (m \boxdot \alpha) \boxdot \beta = m \boxdot (\alpha * \beta), \qquad m \in M,\ \alpha, \beta \in \mathfrak{Q}.$$

A **$\mathfrak{Q}$-linear map** $f : (M, \boxdot) \to (N, \boxdot')$ is a join preserving map such that $f(m \boxdot \alpha) = f(m) \boxdot' \alpha$ for all $m \in M$ and $\alpha \in \mathfrak{Q}$.

Theorem

There is an isomorphism of categories

$$\mathbf{Mod}_r(\mathfrak{Q}) \cong \mathbf{Sup}(\mathfrak{Q}^{op}),$$

where $\mathbf{Mod}_r(\mathfrak{Q})$ *denotes the category of right $\mathfrak{Q}$-modules and $\mathfrak{Q}$-linear maps.*

More precisely, given a right $\mathfrak{Q}$-module $(M, \boxdot)$, there is a unique $\mathfrak{Q}^{op}$-category structure (M, p) such that (M, p) is skeletal, cocomplete, the underlying preorder associated to (M, p) is the original order from M and satisfies

$$p(x \boxdot \alpha, y) = \alpha \searrow p(x, y) \qquad \forall x, y \in M,\ \alpha \in \mathfrak{Q}.$$

In this situation, the relation

$$\mathbf{Sup}_{(M,p)}(f) = \bigvee_{x \in M} x \boxdot f(x)$$

holds for any contravariant $\mathfrak{Q}^{op}$-presheaf f.

Background: The free right $\mathfrak{Q}$-module

Let X be a set and $P(X)$ be the power set of X — i.e. the free complete lattice on X. Now, let $\mathfrak{Q}^X$ be provided with the pointwise order induced by the order on $\mathfrak{Q}$ and with the right multiplication on $\mathfrak{Q}^X$ as right action — i.e.

$$(f * \alpha)(x) = f(x) * \alpha, \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X.$$

Then there exists a right $\mathfrak{Q}$-module isomorphism $\mathfrak{Q}^X \cong P(X) \otimes \mathfrak{Q}$. Since $P(X) \otimes \mathfrak{Q}$ is the free right $\mathfrak{Q}$-module on $P(X)$ and the power set functor is left adjoint to the forgetful functor **Sup** $\to$ **Set**, we conclude that $(\mathfrak{Q}^X, *)$ is the **free right $\mathfrak{Q}$-module on X**. The corresponding $\mathfrak{Q}^{op}$-enriched category structure d of $(\mathfrak{Q}^X, *)$ attains the form:

$$d(f_1, f_2) = \bigwedge_{x \in X} f_1(x) \searrow f_2(x), \qquad f_1, f_2 \in \mathfrak{Q}^X.$$

Because **Sup** is self-dual and $\mathfrak{Q}^X$ is also a left $\mathfrak{Q}$-module w.r.t. the left multiplication — i.e.

$$(\alpha * f)(x) = \alpha * f(x), \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X,$$

there is right action $\boxdot^{op}$ on the dual lattice of $\mathfrak{Q}^X$ is determined by:

$$(f \boxdot^{op} \alpha)(x) = \alpha \searrow f(x), \qquad f \in \mathfrak{Q}^X,\ \alpha \in \mathfrak{Q},\ x \in X.$$

Now let $\leq^{op}$ be the dual order of the pointwise order on $\mathfrak{Q}^X$. Then the associated $\mathfrak{Q}^{op}$-enriched category $d^\dagger$ of $((\mathfrak{Q}^X, \leq^{op}), \boxdot^{op})$ attains the form:

$$d^\dagger(f_1, f_2) = \bigwedge_{x \in X} f_1(x) \swarrow f_2(x), \qquad f_1, f_2 \in \mathfrak{Q}^X.$$

$\mathfrak{Q}$-enriched topologies

We fix an isotone binary operation $\diamond$ on $\mathfrak{Q}$. Then $(\mathfrak{Q}, \diamond)$ is called a **quasi-magma** on $\mathfrak{Q}$ if it satisfies the following condition for all $\alpha, \beta, \gamma \in \mathfrak{Q}$:

$$\alpha * (\beta \diamond \gamma) \leq (\alpha * \beta) \diamond \gamma \quad \text{and} \quad (\alpha \diamond \beta) * \gamma \leq \alpha \diamond (\beta * \gamma).$$

Example

- $\diamond = *$ is always a quasi-magma.
- If $\mathfrak{Q}$ is integral, then $\diamond = \wedge$ is a quasi-magma.

A **$\mathfrak{Q}$-enriched topology** on a set X is a right $\mathfrak{Q}$-submodule $\mathcal{T}$ of the free right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ satisfying the following axioms:

1. $\underline{\top}$ is an element of $\mathcal{T}$.
2. If $f_1, f_2 \in \mathcal{T}$, then $f_1 \diamond f_2 \in \mathcal{T}$, (where $f_1 \diamond f_2$ is defined pointwisely).

The pair $(X, \mathcal{T})$ is called a **$\mathfrak{Q}$-enriched topological space** and each $f \in \mathcal{T}$ is an **open $\mathfrak{Q}$-presheaf on X.**

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces. A map $X \xrightarrow{\varphi} Y$ is **$\mathfrak{Q}$-continuous** if $f \circ \varphi \in \mathcal{T}_X$ for all $f \in \mathcal{T}_Y$.

Note that if $\mathcal{S}$ is a subbase of $\mathcal{T}_Y$ then φ is $\mathfrak{Q}$-continuous if and only if $f \circ \varphi \in \mathcal{T}_X$ for all $f \in \mathcal{S}$.

Obviously, $\mathfrak{Q}$-topological spaces and $\mathfrak{Q}$-continuous maps form a category **Top**$(\mathfrak{Q}, \diamond)$ which is topological over **Set**.

Why study $\mathfrak{Q}$-enriched topologies?

Many-valued topology.

1. **Probabilistic topologies.** When $\mathfrak{Q}$ is a complete MV-algebra and $\diamond = \wedge$, probabilistic topologies associated to probabilistic metric spaces are $\mathfrak{Q}$-topological spaces.

   U. Höhle,
   Probabilistic topologies induced by L-fuzzy uniformities,
   Manuscripta Math. **38**, 289–323 (1982).

2. **Goguen's $\mathfrak{Q}$-spaces.** When $\diamond = *$, Goguen's $\mathfrak{Q}$-spaces with all constant maps to be assumed open are also $\mathfrak{Q}$-topological spaces.

   J.A. Goguen,
   The fuzzy Tychonoff theorem,
   J. Math. Anal. Appl. **43**, 734–742 (1973).

3. **Others:** When $\mathfrak{Q} = ([0,1], *_{\text{Ł}})$ and $\diamond$ is the arithmetic mean, we obtain topologies on spaces of Borel probability measures...

Why study $\mathfrak{Q}$-enriched topologies?

Quantale-valued topological spaces & monoidal topology.

H. Lai, and W. Tholen,
Quantale-valued topological spaces via closure and convergence,
Topol. Appl. **230**, 599–620 (2017)

A **$\mathfrak{Q}$-valued topological space** is a pair (X, c) where X is a set and $c\colon \mathcal{P}(X) \to \mathfrak{Q}^X$ such that for all $A, B \subseteq X$ and $x \in X$, the following conditions hold:

- $c(\varnothing)(x) = \bot$,
- $x \in A$ implies $e \leq c(A)(x)$,
- $c(A)(x) \vee c(B)(x) = c(A \cup B)(x)$,
- $(\bigwedge_{y \in B} c(A)(y)) * c(B)(x) \leq c(A)(x)$.

Morphisms $(X, c) \to (Y, c')$ are contractive maps $X \to Y$.

If the underlying lattice of $\mathfrak{Q}$ is completely distributive, then the category of $\mathfrak{Q}$-valued topological spaces is isomorphic to the category of lax algebras of the lax extension of the ultrafilter monad into the category of $\mathfrak{Q}$-valued relations.

Proposition

Let $\mathfrak{Q}$ be an integral quantale with a dualizing element whose underlying lattice is completely distributive, and set $\diamond = \wedge$. Then the category of $\mathfrak{Q}$-valued topological spaces is a coreflective subcategory of the category of $\mathfrak{Q}$-enriched topological spaces.

For the case of MV-algebras, $\mathfrak{Q}$-valued spaces correspond precisely to a well-understood full subcategory of $\textbf{Top}(\mathfrak{Q}, \diamond)$:

Proposition

*Let $\mathfrak{Q}$ be a complete MV-algebra whose underlying lattice is completely distributive. Then, the category $(\beta, \mathfrak{Q})$−**Alg** is equivalent to the full subcategory of $\textbf{Top}(\mathfrak{Q}, \wedge)$ consisting of $\mathfrak{Q}$-enriched topological spaces that satisfy*

$$\alpha \to f \in \mathcal{T}, \qquad \text{for every } f \in \mathcal{T},\ \alpha \in \mathfrak{Q}. \tag{$\dagger$}$$

$\mathfrak{Q}$-enriched topologies satisfying (†) have particularly simple expressions:

$$\mathcal{T} = \{ f \in \mathfrak{Q}^X \mid f = \bigvee_{b(x)=e} d(b, f) \}.$$

Unfortunately, this does not cover the case $\mathfrak{Q} = [0, \infty]$ (the Lawvere quantale). However, it can be shown that approach spaces are also $\mathfrak{Q}$-enriched topological spaces when $\mathfrak{Q} = ([0, 1], *_Ł)$ and $\diamond = \wedge$. For more details, see

J. Gutiérrez García, U. Höhle, T. Kubiak,
Basic concepts of quantale-enriched topologies,
Appl. Categ. Struct. **29** (2021), 983–1003.

The topologization of ideal lattices of noncommutative C^*-algebras

If A is a commutative C^*-algebra, then the set $\mathbb{R}(A)$ of right-sided (i.e. two-sided) closed ideals of A is a spatial frame which is isomorphic to the Gelfand topology on the set $\mathbb{P}(A)$ of maximal (right) ideals of A.

Suppose now A is a non-commutative C^*-algebra and denote again by $\mathbb{R}(A)$ the set of right-sided ideals of A. Then $\mathbb{R}(A)$ is clearly closed under intersections, and it is easy to show that the join is the closure of the linear hull of the union. Moreover, we have the ideal multiplication

$$I \star J = \mathrm{cl}(\mathrm{lin.\ hull}(\{x \cdot y \mid x \in I,\ y \in J\}))$$

which preserves arbitrary joins in each variable.

Then, $(\mathbb{R}(A), \star)$ is a quantale which is:

- non-commutative,
- non-unital
- idempotent,
- right-sided (i.e. $\alpha * \top \leq \alpha$ for all $\alpha \in \mathfrak{Q}$),
- every maximal element is prime ($\beta * \gamma \leq \alpha \implies \beta * \top \leq \alpha$ or $\top * \gamma \leq \alpha$),
- every element is a meet of maximal elements, and so it is spatial.

Question: can we find a ($\mathfrak{Q}$-enriched) topological space $(\mathbb{P}(A), \mathcal{T})$ and a quantale isomorphism $\mathcal{T} \cong \mathbb{R}(A)$?

In what follows we choose $\diamond = *$. Then, since $\mathcal{T}$ must be a subquantale of $\mathfrak{Q}^{\mathbb{P}(A)}$ (closed additionally under $\underline{\top}$), it follows that $\mathfrak{Q}$ must be non-commutative (and unital!). Hence we need at least 4 elements.

It is natural to consider the idempotent and right-sided quantale C_4^r on the 4-chain $\{\bot < a < e < \top\}$ given by the multiplication table

$*$	$\bot$	a	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a
e	$\bot$	a	e	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$

Its subquantale $C_3^r := \{\bot < a < \top\}$ is the unique non-commutative, idempotent and right-sided quantale with three elements.

It is well-known that if $\mathfrak{Q}$ is a right-sided and idempotent quantale, then prime elements $p \in \mathfrak{Q}$ correspond to a strong quantale homomorphisms $h_p \colon \mathfrak{Q} \longrightarrow C_3^r$ such that $p = \bigvee\{\, x \in \mathfrak{Q} \mid h_p(x) \leq a \,\}$. If in $\mathfrak{Q}$ spatial, then we can identify every $x \in \mathfrak{Q}$ with a map $\mathbb{A}_x \colon \mathrm{Prime}(\mathfrak{Q}) \to C_3^r$ given by $\mathbb{A}_x(p) = h_p(x)$.

Then, the set

$$\mathcal{T} := \{\mathbb{A}_I \colon \mathbb{P}(A) \to C_4^r \mid I \in \mathbb{R}(A)\}$$

is in bijection with $\mathbb{R}(A)$. It is easy to verify that the relations

$$\bigvee_{i \in A} \mathbb{A}_{I_i} = \mathbb{A}_{\bigvee_{i \in I} I_i}, \quad \mathbb{A}_I * \mathbb{A}_J = \mathbb{A}_{I \star J}, \quad \mathbb{A}_A = \underline{\top},$$

$$\mathbb{A}_I * \alpha = \mathbb{A}_I$$

hold for any $\{I_i\}_{i \in I} \subseteq \mathbb{R}(A)$, $I, J \in \mathbb{R}(A)$, $\alpha \in C_4^r$. Hence $\mathcal{T}$ is a C_4^r-enriched topology on $\mathbb{P}(A)$ isomorphic to $\mathbb{R}(A)$.

Remark

If A is commutative, then the change of base determined by $\mathbf{2} \hookrightarrow C_4^r$ yields the traditional Gelfand topology.

J. Gutiérrez García, and U. Höhle,
Right algebras in Sup and the topological representation of semi-unital and semi-integral quantales, revisited,
Mathematica Slovaca **74** (2) (2024), 261–280

Interior and adherence operators

A **$\mathfrak{Q}$-interior operator** on a set X is a $\mathfrak{Q}$-coclosure operator $\mathcal{I}$ on $(\mathfrak{Q}^X, d)$ satisfying the following additional properties:

(Int1) $\mathcal{I}(\top) = \top$,

(Int2) If $f_1, f_2 \in \mathfrak{Q}^X$, then $\mathcal{I}(f_1) \diamond \mathcal{I}(f_2) \leq \mathcal{I}(f_1 \diamond f_2)$.

Since $\mathfrak{Q}$-coclosure operators on $(\mathfrak{Q}^X, d)$ and right $\mathfrak{Q}$-submodules of $(\mathfrak{Q}^X, *)$ are equivalent concepts, it is easily seen that every $\mathfrak{Q}$-interior operator $\mathcal{I}$ on X can be identified with a $\mathfrak{Q}$-enriched topology $\mathcal{T}$ on X and vice versa.

Every $\mathfrak{Q}$-interior operator $\mathcal{I}$ on X induces a $\mathfrak{Q}$-coclosure operator $\mathcal{A}_{\mathcal{I}}$ on $\left(\mathfrak{Q}^X, d^{\dagger}\right)$. It is given by the expression

$$\mathcal{A}_{\mathcal{I}}(f)(x) = \bigwedge_{g \in \mathfrak{Q}^X} \Big(\big(\bigvee_{y \in X} (f(y) * g(y)) \big) \swarrow \mathcal{I}(g)(x) \Big), \qquad x \in X, f \in \mathfrak{Q}^X.$$

Lower separation

Definition

A $\mathfrak{Q}$-enriched topological space $(X, \mathcal{T})$ is said to be

1. T_0, if for each $x, y \in X$ with $x \neq y$ there exists some $f \in \mathcal{T}$ such that $f(x) \not\leq f(y)$ or $f(y) \not\leq f(x)$.
2. T_1, if for each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ with $f_1(x) \not\leq f_1(y)$ *and* $f_2 \in \mathcal{T}$ with $f_2(y) \not\leq f_2(x)$.

Let $\mathfrak{Q} = (\mathfrak{Q}, *, e, ')$ be an involutive quantale. The **specialization $\mathfrak{Q}^{op}$-enriched category** p_s of a $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is determined by

$$p_s(x, y) = \mathcal{A}_{\mathcal{I}}(1_{\{y\}})(x)' = \bigwedge_{f \in \mathcal{T}} f'(x) \searrow f'(y), \qquad x, y \in X.$$

H. Lai, D. Zhang,
Fuzzy preorder and fuzzy topology,
Fuzzy Sets and Systems (14) **157** (2006), 1865–1885.

1. $(X, \mathcal{T})$ is T_0 if and only if the specialization $\mathfrak{Q}^{op}$-enriched category is skeletal.
2. $(X, \mathcal{T})$ is T_1 if and only if the specialization $\mathfrak{Q}^{op}$-enriched category is discrete.

The Hausdorff axiom

A $\mathfrak{Q}$-enriched topological space is **Hausdorff** (or T_2) if for each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ with

$$f_1(x) \diamond f_2(y) \not\leq \bigvee_{z \in X} (f_1(z) \diamond f_2(z)) \quad \text{or} \quad f_2(y) \diamond f_1(x) \not\leq \bigvee_{z \in X} (f_2(z) \diamond f_1(z)).$$

We say that a subset A of X is **dense** in $(X, \mathcal{T})$ if every $x \in X$ is an adherent point of 1_A — i.e. if $\underline{e} \leq \mathcal{A}_{\mathcal{I}}(1_A)$.

Proposition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-enriched topological spaces and A be a dense subset in $(X, \mathcal{T}_X)$. If $(Y, \mathcal{T}_Y)$ is Hausdorff separated and the maps $(X, \mathcal{T}_X) \xrightarrow{\psi, \varphi} (Y, \mathcal{T}_Y)$ are $\mathfrak{Q}$-continuous such that their restrictions to A coincide, then $\psi = \varphi$.

The regularity axiom

Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-enriched topological space. For every open $\mathfrak{Q}$-presheaf $f \in \mathcal{T}$ consider the $\mathfrak{Q}$-presheaf $F_f \colon \mathcal{T} \to \mathfrak{Q}$ given by

$$F_f(g) = d(\mathcal{A}(g), f).$$

We say that $(X, \mathcal{T})$ is

- **regular** if for any open $\mathfrak{Q}$-presheaf $f \in \mathcal{T}$,

  $$f = \mathbf{Sup}_{(\mathcal{T},d)}(F_f).$$

- **weakly regular** if the set

  $$\left\{ f \in \mathcal{T} \mid f = \mathbf{Sup}_{(\mathcal{T},d)}(F_f) \right\}$$

  is a subbase for $\mathcal{T}$.

When $\mathfrak{Q} = \mathbf{2}$, regularity and weak regularity coincide with the usual separation axiom.

More generally,

Proposition

If $\mathfrak{Q}$ is a frame (i.e. $* = \wedge$) and $\diamond = \wedge$, then regularity and weak regularity are equivalent.

The next example presents a weakly regular $\mathfrak{Q}$-topological space being *not* regular.

Example

On the 3-chain $C_3 = \{\perp, a, \top\}$ consider the 3-valued *MV*-algebra $\mathfrak{Q} = (C_3, *)$ and $\diamond = *$. Let $\mathcal{T}$ be the $\mathfrak{Q}$-topology on C_3 generated by the subbase $\mathcal{S} = \{\mathrm{id}_{C_3}, \mathrm{id}_{C_3} \to \perp\}$. It is not difficult to show that $(C_3, \mathcal{T})$ is weakly regular but not regular. Since $(\mathfrak{Q}, \boxdot = *)$ is a projective right $\mathfrak{Q}$-module and $\mathcal{T}$ coincides with the interval topology on C_3, we will extend this situation later.

For the remaining part of this talk we will work in the framework of the quasi-magma $(\mathfrak{Q}, *)$ — i.e. $\diamond = *$.

Proposition

Let $\mathfrak{Q}$ be quantale with a dualizing element. Then in any weakly regular $\mathfrak{Q}$-enriched topological space,

$$T_0 \iff T_1 \iff T_2.$$

The principle of continuous extension

A classical result from

N. Bourbaki, J. Dieudonné,
Note de tératopologie. II,
Revue Scientifique (Revue Rose) **77** (1939), 180–181.

provides conditions for a continuous function defined in a dense subspace of a regular space to admit a unique continuous extension.

In order to obtain a $\mathfrak{Q}$-enriched generalization, we first develop a few concepts regarding the convergence theory of $\mathfrak{Q}$-enriched spaces.

A $\mathfrak{Q}$-**enriched filter** on a set X is a covariant $\mathfrak{Q}$-presheaf ω on $(\mathfrak{Q}^X, d)$ satisfying the following properties for all $f_1, f_2 \in \mathfrak{Q}^X$:

1. $\omega(\underline{\top}) = \top$,
2. $\omega(f) \diamond \omega(g) \leq \omega(f \diamond g)$,
3. $\omega(f) \leq \bigvee f(X)$.

Given a $\mathfrak{Q}$-enriched topological space $(X, \mathcal{T})$ and $x \in X$, the covariant $\mathfrak{Q}$-presheaf ν_x on $(\mathfrak{Q}^X, d)$ given by

$$\nu_x(f) = \mathcal{I}(f)(x), \qquad f \in \mathfrak{Q}^X$$

is a $\mathfrak{Q}$-filter, namely the **$\mathfrak{Q}$-neighborhood filter at x**.

An element $x \in X$ is called a **limit point** of a $\mathfrak{Q}$-filter ω in $(X, \mathcal{T})$ if $\nu_x(f) \leq \omega(f)$ for all $f \in \mathfrak{Q}^X$.

Remark

In a Hausdorff $\mathfrak{Q}$-enriched topological space limit points are unique.

A map $(X, \mathcal{T}_X) \xrightarrow{\varphi} (Y, \mathcal{T}_Y)$ between $\mathfrak{Q}$-enriched topological spaces is $\mathfrak{Q}$-continuous if and only if for each $x \in X$ and each $\mathfrak{Q}$-filter ω on X converging to x the image $\mathfrak{Q}$-filter $\varphi(\omega)$ converges to $\varphi(x)$, where

$$\big(\varphi(\omega)\big)(g) = \omega(g \circ \varphi), \qquad g \in \mathfrak{Q}^Y.$$

Every $\mathfrak{Q}$-presheaf f on A can be identified with the $\mathfrak{Q}$-presheaf f_X on X determined by:

$$f_X(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \top, & \text{if } x \notin A, \end{cases} \qquad x \in X.$$

Further, let ω be a $\mathfrak{Q}$-filter on X. The **trace** of ω on A is the covariant $\mathfrak{Q}$-presheaf ω_A on $\left(\mathfrak{Q}^A, d\right)$ defined by:

$$\omega_A(f) := \omega(f_X), \qquad f \in \mathfrak{Q}^A.$$

It is not difficult to show that ω_A is a filter on A if and only if A is dense.

Theorem (Principle of $\mathfrak{Q}$-continuous extension)

Let $\mathfrak{Q}$ be quantale with a dualizing element, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be $\mathfrak{Q}$-topological spaces, A be a dense subset in $(X, \mathcal{T}_X)$ and $\iota\colon A \hookrightarrow X$ be the inclusion map. Further, let $(Y, \mathcal{T}_Y)$ be T_0 and weakly regular and $\psi\colon A \to Y$ be a $\mathfrak{Q}$-continuous map w.r.t. the initial $\mathfrak{Q}$-topology on A induced by $\mathcal{T}_X$. Then the following assertions are equivalent:

1. *ψ has a unique $\mathfrak{Q}$-continuous extension to X — i.e. there exists a unique $\mathfrak{Q}$-continuous map $X \xrightarrow{\varphi} Y$ making the following diagram commutative:*

$$\begin{array}{ccc} A & \xhookrightarrow{\ \iota\ } & X \\ & {\scriptstyle\psi}\searrow & \big\downarrow{\scriptstyle\varphi} \\ & & Y \end{array}$$

2. *There exists a map $X \xrightarrow{\varphi} Y$ such that for all $x \in X$ the point $\varphi(x)$ is a limit point of $\psi\big((\nu_x)_A\big)$, where $\psi\big((\nu_x)_A\big)$ denotes the image of the trace of the $\mathfrak{Q}$-neighborhood filter ν_x on A under ψ.*

The proof of this result relies essentially on the assumption $\diamond = *$. We therefore propose the following:

Open Problem. Let $\mathfrak{Q}$ be an integral quantale. Does the principle of $\mathfrak{Q}$-continuous extension hold in the case of the quasi-magma $(\mathfrak{Q}, \wedge)$?

B. Banaschewski,
Extension by continuity in pointfree topology,
Appl. Categ. Struct. **8** (2000), 475–486.

Projective right $\mathfrak{Q}$-modules and the interval topology

I. Stubbe,
Towards “dynamic domains”. Totally continuous cocomplete categories,
Theoret. Comput. Sci. **373**, 142–160 (2007).

A cocomplete $\mathfrak{Q}$-enriched category (X, p) is said to be (constructively) **$\mathfrak{Q}$-enriched completely distributive** if the left adjoint of the Yoneda embedding $\mathbf{y}_{(X,p)} : (X, p) \to \mathbb{P}(X, p)$, that is $\mathbf{Sup}_{(X,p)} : \mathbb{P}(X, p) \to (X, p)$, has a further left adjoint $\lhd_{(X,p)} : (X, p) \to \mathbb{P}(X, p)$.

Proposition

A cocomplete $\mathfrak{Q}$-enriched category (X, p) is $\mathfrak{Q}$-enriched completely distributive if and only if it is a projective object in $\mathbf{Sup}(\mathfrak{Q})$.

Example

For any (unital) quantale $\mathfrak{Q}$, its right multiplication determines a right $\mathfrak{Q}$-module structure on $\mathfrak{Q}$ which is projective.

Let $\mathfrak{Q}$ be a commutative Girard quantale and M be a projective $\mathfrak{Q}$-module in **Sup** with associated $\mathfrak{Q}^{op}$- enriched category (M, p).

Choose a dualizing element $\delta \in \mathfrak{Q}$ and define the **interval $\mathfrak{Q}$-topology** $\mathcal{T}_I$ on M generated by the following subbase:

$$\{\, p(\_, m) \searrow \delta) \mid m \in M \,\} \cup \{\, p(n, \_) \searrow \delta \mid n \in M \,\}.$$

Theorem

Let $\mathfrak{Q}$ be a commutative Girard quantale and M be a projective $\mathfrak{Q}$-module in **Sup**. *Then the interval topology on M is a T_0 and weakly regular $\mathfrak{Q}$-enriched topology.*

Future directions

- Develop the theory of compactness by using flat $\mathfrak{Q}$-enriched ideals (compact Hausdorff $\mathfrak{Q}$-enriched spaces, $\mathfrak{Q}$-enriched Hofmann–Mislove Theorem, etc.).

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- Comparison of distinct separation axioms with those occurring in monoidal topology.

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Muito obrigado!