# Lelek fan as an inverse limit of Cantor fans

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Joint work with Goran Erceg, Judy Kennedy, Chris Mouron and Van Nall

**Definition**

A continuum is a non-empty compact connected metric space.

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A subcontinuum is a subspace of a continuum, which is itself a continuum.

### Definition

Let $X$ be a continuum. We say that $X$ is a Cantor fan, if $X$ is homeomorphic to the continuum $\bigcup_{c \in C} A_c$, where $C \subseteq [0,1]$ is a Cantor set and for each $c \in C$, $A_c$ is the convex segment in the plane from $(0,0)$ to $(c,-1)$.

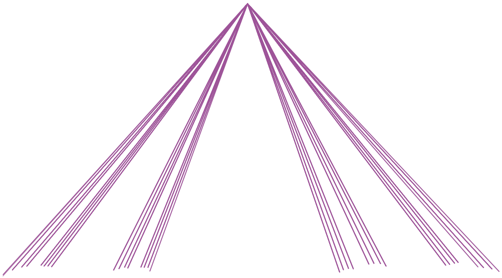


Figure: The Cantor fan

## Definition

Let $X$ be a Cantor fan and let $Y$ be a subcontinuum of $X$. A point $x \in Y$ is called an end-point of the continuum $Y$, if for every arc $A$ in $Y$ that contains $x$, $x$ is an end-point of $A$.

The set of all end-points of $Y$ will be denoted by $E(Y)$.

## Definition

Let $X$ be a Cantor fan and let $Y$ be a subcontinuum of $X$. We say that $Y$ is a Lelek fan, if $\mathrm{Cl}(E(Y)) = Y$.

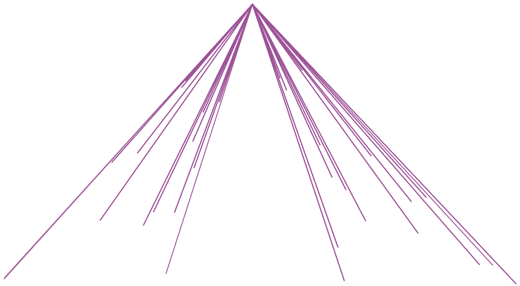


Figure: The Lelek fan

## Definition

Suppose $X$ is a compact metric space. If $f : X \to X$ is a continuous function, the inverse limit space generated by $f$ is the subspace

$$\varprojlim(X, f) = \left\{(x_1, x_2, x_3, \dots) \in \prod_{i=1}^{\infty} X \mid \text{ for each } i, x_i = f(x_{i+1})\right\}$$

of the topological product $\prod_{i=1}^{\infty} X$.

### Definition

Let $(X, f)$ be a dynamical system. We say that $(X, f)$ is transitive, if for all non-empty open sets $U$ and $V$ in $X$, there is a non-negative integer $n$ such that $f^n(U) \cap V \neq \emptyset$.

## Definition

Let $X$ be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on $X$. If $F$ is closed in $X \times X$, then we say that $F$ is a closed relation on $X$.

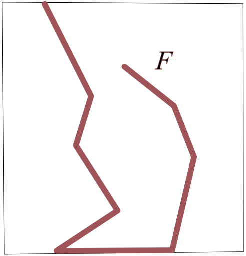



Figure: A closed relation $F$ on $I = [0, 1]$

## Definition

Let $X$ be a non-empty compact metric space and let $F$ be a closed relation on $X$. Then we call

$$X_F^+ = \left\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \mid \text{ for each } i, (x_i, x_{i+1}) \in F\right\}$$

the Mahavier product of $F$, and we call

$$X_F = \left\{(\ldots, x_{-1}, x_0; x_1, x_2, \ldots) \in \prod_{i=-\infty}^{\infty} X \mid \text{ for each } i, (x_i, x_{i+1}) \in F\right\}$$

the two-sided Mahavier product of $F$.

$X_F^+$ and $X_F$ are subspaces of the topological products $\prod_{i=1}^{\infty} X$ and $\prod_{i=-\infty}^{\infty} X$, respectively.

## Definition

Let $X$ be a non-empty compact metric space and let $F$ be a closed relation on $X$.

The function $\sigma_F^+ : X_F^+ \to X_F^+$, defined by

$$\sigma_F^+(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, x_4, \ldots) \in X_F^+$, is called the shift map on $X_F^+$.

**Theorem (I.B., G. Erceg, J. Kennedy, C. Mouron, V. Nall)**

Let $X$ be a compact metric space and let $F$ be a closed relation on $X$. Then

$$\varprojlim(X_F^+, \sigma_F^+) \text{ is homeomorphic to } X_F.$$

## Theorem

There is a transitive mapping $f$ on a Cantor fan $X$ such that $\varprojlim(X, f)$ is a Lelek fan.

PROOF: We use $I$ to denote $I = [0,1]$ and we use $H$ to denote

$$H = \left\{\left(x, \sqrt{x}\right) \mid x \in [0,1]\right\} \cup \left\{\left(x, \frac{1}{2}x^3\right) \mid x \in [0,1]\right\}.$$

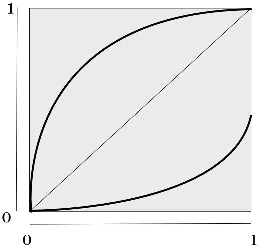
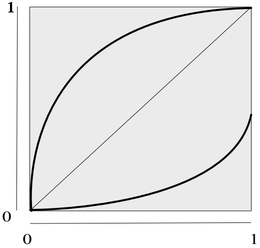



Figure: The closed relation $H$ on $I = [0,1]$

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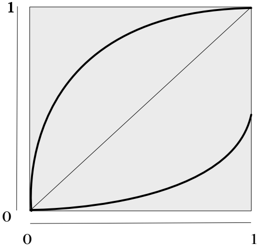
Then prove:
$I_H^+$ is a Cantor fan

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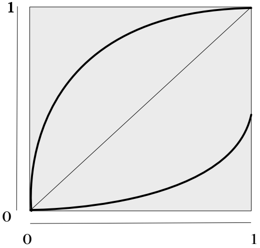


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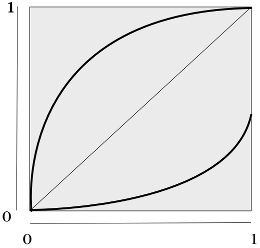
$I_H^+$ is a Cantor fan and $I_H$ is a Lelek fan. Also, $\sigma_H^+ : I_H^+ \to I_H^+$ is transitive.

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$I_H$ is homeomorphic to $\varprojlim(I_H^+, \sigma_H^+)$.

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There is a transitive mapping $f$ on a Cantor fan $X$ such that $\varprojlim(X, f)$ is a Lelek fan.

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**Problem**

Is there a transitive mapping $f$ on a Lelek fan $X$ such that $\varprojlim(X,f)$ is a Cantor fan?

THANK YOU!