Cardinal inequalities for Urysohn spaces involving variations of the almost Lindelf degree

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Theorem: [Pospišil–1937]

If X is a Hausdorff space, then $|X| \leq d(X)^{\chi(X)}$.

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Theorem: [Arhangel′skiĭ–1971, Šapirovskiĭ–1974]

If X is a Hausdorff space, then $|X| \leq 2^{t(X)\psi(X)L(X)}$.

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Note: $\psi(X) \leq \psi_c(X)$ for every Hausdorff space X.

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Note: $aL(X) \leq aL_c(X) \leq L(X)$ for every space X.

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The θ-*closure* of a set A in a space X is the set

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A is called θ-dense in X if $\mathrm{cl}_\theta(A) = X$.

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Note: $\psi(x) \leq \psi_c(X) \leq \kappa(X) \leq \chi(X)$ for every Hausdorff space X.

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Note: Kočinac' inequality could be considered as an attempt to find, for the class of Urysohn spaces, the counterpart of Bella and Cammaroto inequality that if X is a Hausdorff space, then $|X| \leq d(X)^{t(X)\psi_c(X)}$.

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Note 1: If V is open, then $\mathrm{cl}_\theta(V) = \mathrm{cl}(V)$ and therefore $\mathrm{cl}_\theta(\mathrm{cl}_\theta(V)) = \mathrm{cl}_\theta(\mathrm{cl}(V))$. This explains our notation $\psi_{\theta^2}(X)$.

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Note 2: $\psi(x) \leq \psi_c(X) \leq \psi_{\theta^2}(X) \leq \kappa(X) \leq \chi(X)$ for every Urysohn space X.

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Corollary: If X is a Urysohn H-closed space then $\psi_{\theta^2}(X) = \psi_c(X)$.

Therefore, the former corollary extends Kočinac' inequality to all Urysohn spaces.

A generalization of $|X| \leq 2^{k(X)aL(X)}$

The following theorem generalizes the theorem that if X is a Urysohn space, then $|X| \leq 2^{k(X)aL(X)}$ and therefore Bella and Cammaroto inequality $|X| \leq 2^{\chi(X)aL(X)}$.

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The following theorem generalizes the theorem that if X is a Urysohn space, then $|X| \leq 2^{k(X)aL(X)}$ and therefore Bella and Cammaroto inequality $|X| \leq 2^{\chi(X)aL(X)}$.

Theorem:

For every Urysohn space X, $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$.

The proof uses the "closure" method and the following inequality from the previous theorem:
$|\mathrm{cl}_\theta(A)| \leq |A|^{t_\theta(X)\psi_{\theta^2}(X)}$.

Definition of $S(n)$-spaces

To compare our result $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$ with Bella and Cammaroto–Hodel inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$ we recall the following definition:

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Definition: [Viglino–1971; Porter and Votaw–1973]
Let X be a topological space, $A \subset X$ and $n \in \mathbb{N}^+$. A point $x \in X$ is *$S(n)$-separated from A* if there exist open sets U_i, $i = 1, 2, ..., n$ such that $x \in U_1$, $\overline{U}_i \subset U_{i+1}$ for $i = 1, 2, ..., n-1$ and $\overline{U}_n \cap A = \varnothing$;

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X is an $S(n)$*-space* if every two distinct points in X are $S(n)$-separated.

It follows directly from the above definitions that the $S(1)$-spaces are exactly the Hausdorff spaces and the $S(2)$-spaces are exactly the Urysohn spaces.

A generalization of $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$

Lemma: If X is an $S(3)$-space, then $\psi_{\theta^2}(X) \leq \psi(X)aL_c(X)$.

A generalization of $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$

Lemma: If X is an $S(3)$-space, then $\psi_{\theta^2}(X) \leq \psi(X)aL_c(X)$.

Therefore, for the class of $S(3)$-spaces X, the inequality

$$|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$$

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We recall that in 2014 Cammaroro, Catalioto and Porter showed that even for H-closed Urysohn spaces X it is possible $t_\theta(X) < t(X)$, $t_\theta(X) > t(X)$, or $t_\theta(X) = t(X)$.

A generalization of $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$

Definition: [Cammaroto and Kočinac – 1993] For a topological space X, $t_{\theta_1}(X)$ is the smallest infinite cardinal κ such that for every $A \subset X$ and every $x \in \mathrm{cl}(X)$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \mathrm{cl}_\theta(B)$.

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Note: $t_{\theta_1}(X) \leq t_\theta(X)$ and $t_{\theta_1}(X) \leq t(X)$ for every space X.

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Note: $t_{\theta_1}(X) \leq t_\theta(X)$ and $t_{\theta_1}(X) \leq t(X)$ for every space X.

Therefore, if we want to get a stronger inequality it is better to try to replace $t(X)$ with $t_{\theta_1}(X)$.

A generalization of $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$

Theorem:

If X is a Urysohn space, then $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X)}$.

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Since for $S(3)$-spaces X we have $\psi_{\theta^2}(X) \leq \psi(X)aL_c(X)$, the inequality in the former theorem better approximate the cardinality of $S(3)$-spaces than Bella and Cammaroto–Hodel inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$.

A generalization of $|X| \leq 2^{t(X)\psi(X)L(X)}$

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Since for Urysohn spaces X we have $\psi_{\theta^2}(X) \leq \psi(X)L(X)$, the inequality

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better approximate the cardinality of Urysohn spaces X than Arhangel′skiĭ–Šapirovskiĭ's inequality

$$|X| \leq 2^{t(X)\psi(X)L(X)}.$$

Another generalization of $|X| \leq 2^{t(X)\psi(X)L(X)}$

Definition: [Basile, Bonanzinga, Carlson – 2018] The *θ-almost Lindelöf degree of a subset Y of a space X*, is $\theta\text{-}aL(Y, X) = \min\{\kappa :$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $\bigcup\{cl_\theta(\mathrm{cl}(V)) : V \in \mathcal{V}'\} = Y\}$.

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The function $\theta\text{-}aL(X, X)$ is called *θ-almost Lindelöf degree of the space X* and is denoted by $\theta\text{-}aL(X)$.

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The θ-almost Lindelöf degree with respect to closed subsets of X is $\theta\text{-}aL_c(X) = \sup\{\theta\text{-}aL(C,X) : C \subseteq X \text{ is closed}\}$.

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The θ-almost Lindelöf degree with respect to θ-closed subsets of X is $\theta\text{-}aL_\theta(X) = \sup\{\theta\text{-}aL(C,X) : C \subseteq X$ is θ-closed$\}$.

A generalization of $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)aL(X)}$

Theorem:

If X is a Urysohn space, then $|X| \leq 2^{t_\theta(X)\psi_{\theta^2}(X)\theta\text{-}aL_\theta(X)}$.

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Theorem: If X is a Urysohn space, then
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For the proof we need to introduce the following concept:

A generalization of $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)aL_c(X)}$

Theorem: If X is a Urysohn space, then $|X| \leq 2^{t_{\theta_1}(X)\psi_{\theta^2}(X)\theta-aL_c(X)}$.

For the proof we need to introduce the following concept:

Definition: For $A \subset X$ and an infinite cardinal κ, let $[A]^{\leq\kappa} = \{B : B \subseteq A, |B| \leq \kappa\}$.

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We define the *θ-κ-closure of A* as $\mathrm{cl}_{\theta_\kappa}(A) = \bigcup_{B \in [A]^{\leq\kappa}} \mathrm{cl}_\theta(B)$.

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Theorem: If X is a Urysohn space, $A \subset X$, and κ is an infinite cardinal, then $|\mathrm{cl}_{\theta\kappa}(A)| \leq |A|^{\kappa\cdot\psi_{\theta^2}(X)}$.

Since for $S(4)$-spaces X we have $\psi_{\theta^2}(X) \leq \psi(X)\theta\text{-}aL_c(X)$, our new inequality better approximate the cardinality of $S(4)$-spaces than Bella and Cammaroto–Hodel inequality $|X| \leq 2^{t(X)\psi(X)aL_c(X)}$.

The end

THANK YOU!