Specification properties in CR-dynamical systems

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Basic notions

- Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. We say that (X, f) is *a dynamical system*.

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  is called *a forward orbit of the point* x.
- The set

$$\left\{x, f(x), f^2(x), f^3(x), \ldots\right\}$$

  is called *a forward orbit set of the point* x.

Basic notions

- Let (X, f) be a dynamical system, let $x \in X$ and let k, ℓ be non-negative integers such that $k \leq \ell$. We say that

$$f^{[k,\ell]}(x) = \left(f^k(x), f^{k+1}(x), f^{k+2}(x), \ldots, f^{\ell}(x)\right)$$

is *a $[k, \ell]$-orbit segment of the point x.*

Specification property

Definition

Let (X, f) be a dynamical system, let n be a positive integer and, for each $j \in \{1, 2, 3, \dots, n\}$, let

- k_j and ℓ_j be non-negative integers such that $k_j \leq \ell_j$
- $x_j \in X$.

We say that the $n - tuple$

$$\mathcal{S} = \left(f^{[k_1, \ell_1]}(x_1), f^{[k_2, \ell_2]}(x_2), f^{[k_3, \ell_3]}(x_3), \dots, f^{[k_n, \ell_n]}(x_n) \right)$$

is *an n-specification* (or just *a specification) in* (X, f).

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$$\mathcal{S} = \left(f^{[k_1,\ell_1]}(x_1), f^{[k_2,\ell_2]}(x_2), f^{[k_3,\ell_3]}(x_3), \ldots, f^{[k_n,\ell_n]}(x_n)\right)$$

be a specification in (X, f). We say that $\mathcal{S}$ is *an N-spaced specification* if

$$k_{j+1} - \ell_j \geq N,$$

for each $j \in \{1, 2, 3, \ldots, n-1\}$.

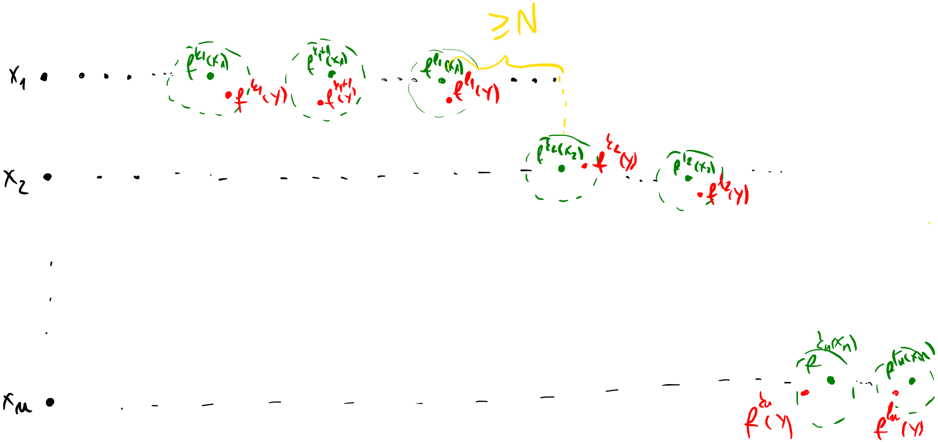
Definition

Let (X, f) be a dynamical system, let N be a positive integer, let $\varepsilon > 0$, let $y \in X$ and let

$$\mathcal{S} = \left(f^{[k_1,\ell_1]}(x_1), f^{[k_2,\ell_2]}(x_2), f^{[k_3,\ell_3]}(x_3), \ldots, f^{[k_n,\ell_n]}(x_n) \right)$$

be an N-spaced specification in (X, f). We say that $\mathcal{S}$ is ε-traced in (X, f) by y if for each $i \in \{1, 2, 3, \ldots, n\}$ and for each $j \in \{k_i, k_i + 1, k_i + 2, \ldots, \ell_i\}$,

$$d(f^j(y), f^j(x_i)) \leq \varepsilon.$$



x1
x2
xn
≥N

Definition

Let (X, f) be a dynamical system. We say that (X, f) *has specification property* if for each $\varepsilon > 0$ there is a positive integer N such that for any N-spaced specification $\mathcal{S}$ in (X, f) there is $y \in X$ such that $\mathcal{S}$ is ε-traced in (X, f) by y.

Shift map and specification property

Let $X_\infty = \varprojlim (X_n, f_n)$ denote the inverse limit of a inverse sequence (X_n, f_n), where $X_n = X$ and $f_n = f$, for every $n \in \mathbb{N}$. We define *the shift map* $\sigma : X_\infty \to X_\infty$ by the rule

$$\sigma (x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

for every $(x_1, x_2, x_3, \ldots) \in X_\infty$.

Theorem (BEJK, 2024.)

Let (X, f) be a dynamical system such that f is a surjection. Then (X, f) has specification property if and only if (X_∞, σ) has specification property.

Initial specification property

- Let (X, f) be a dynamical system and let

$$\mathcal{S} = \left(f^{[k_1,\ell_1]}(x_1), f^{[k_2,\ell_2]}(x_2), f^{[k_3,\ell_3]}(x_3), \ldots, f^{[k_n,\ell_n]}(x_n) \right)$$

be a specification in (X, f). If $k_i = 0$, for each $i \in \{1, 2, 3, \ldots, n\}$, then we say that S is *an initial specification* in (X, f).

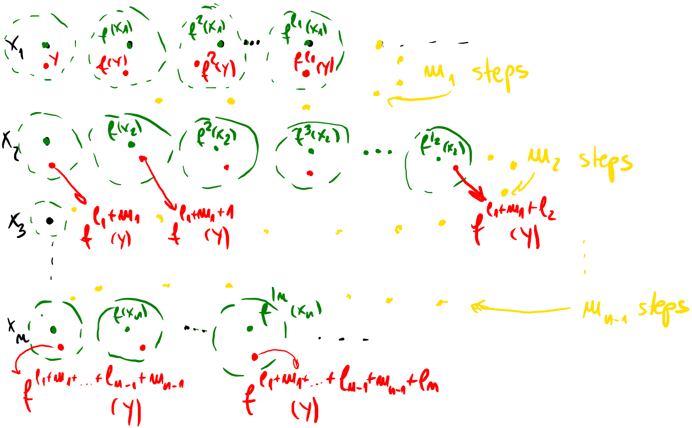
Definition

Let (X, f) be a dynamical system, let $\ell_0 = m_0 = 0$, let $m_1, m_2, m_3, \ldots, m_{n-1}$ be positive integers, let $\varepsilon > 0$, let $y \in X$, and let

$$\mathcal{S} = \left(f^{[0,\ell_1]}(x_1), f^{[0,\ell_2]}(x_2), f^{[0,\ell_3]}(x_3), \ldots, f^{[0,\ell_n]}(x_n) \right)$$

be an initial specification in (X, f). We say that $\mathcal{S}$ is $(\varepsilon, m_1, m_2, m_3, \ldots, m_{n-1})$*-traced in* (X, f) *by* y if for each $i \in \{1, 2, 3, \ldots, n\}$ and for each $j \in \{0, 1, 2, \ldots, \ell_i\}$,

$$d\left(f^{\ell_0+m_0+\ell_1+m_1+\ell_2+m_2+\ldots+\ell_{i-1}+m_{i-1}+j}(y), f^j(x_i) \right) \leq \varepsilon.$$


x1
y
f(x1)
f(y)
f2(x1)
f2(y)
fl1(x1)
fl1(y)
m1 steps
x2
f(x2)
f2(x2)
f3(x2)
fl2(x2)
m2 steps
x3
fl1+m1(y)
fl1+m1+1(y)
fl1+m1+l2(y)
xn
f(xn)
fln(xn)
mn-1 steps
fl1+m1+...+ln-1+mn-1(y)
fl1+m1+...+ln-1+mn-1+ln(y)

Definition

Let (X, f) be a dynamical system. We say that (X, f) *has initial specification property* if for each $\varepsilon > 0$ there is a positive integer N such that

- for each positive integer n,
- for all positive integers $m_1, m_2, m_3, \ldots, m_{n-1}$ such that $m_i \geq N$, for each $i \in \{1, 2, 3, \ldots, n-1\}$,
- for each initial n-specification $\mathcal{S}$,

there is $y \in X$ such that $\mathcal{S}$ is $(\varepsilon, m_1, m_2, m_3, \ldots, m_{n-1})$-traced in (X, f) by y.

SP vs initial SP

Specification property and initial specification property, in general, are not equivalent.

- Let $X = [0, 1]$ and let $f : X \to X$ be defined by

$$f(x) = 0,$$

  for any $x \in X$. Note that (X, f) has specification property but it does not have initial specification property.

Theorem

Let X be a compact metric space and let $f : X \to X$ be a continuous surjection. Then (X, f) has specification property if and only if (X, f) has initial specification property.

- If X is a compact metric space, then 2^X denotes the set of all non-empty closed subsets of X.
- Let X be a compact metric space and let $F \subseteq X \times X$ be a relation on X. If $F \in 2^{X \times X}$, then we say that F is *a closed relation on* X.
- If F is a closed relation on a compact metric space X, then the pair (X, F) is called *a CR-dynamical system.*

- Let (X, F) be a CR-dynamical system, let $x \in X$ and let $k \leq \ell$ be non-negative integers. We use $F^{[k,\ell]}(x)$ to denote

$$F^{[k,\ell]}(x) = \left(F^{k}(x), F^{k+1}(x), F^{k+2}(x), \ldots, F^{\ell}(x)\right)$$

and say that $F^{[k,\ell]}(x)$ is *the* $[k,\ell]$*-orbit segment of the point* x.

Definition

Let (X, F) be a CR-dynamical system, let n be a positive integer and, for each $j \in \{1, 2, \ldots, n\}$, let

- k_j and ℓ_j be non-negative integers such that $k_j \leq \ell_j$
- $x_j \in X$.

We say that the n-tuple (of sets)

$$\left(F^{[k_1,\ell_1]}(x_1), F^{[k_2,\ell_2]}(x_2), F^{[k_3,\ell_3]}(x_3), \ldots, F^{[k_n,\ell_n]}(x_n)\right)$$

is *an n-specification* (or just *a specification*) in (X, F).

Definition

Let (X, F) be a CR-dynamical system and let

$$S = \left(F^{[k_1,\ell_1]}(x_1), F^{[k_2,\ell_2]}(x_2), F^{[k_3,\ell_3]}(x_3), \ldots, F^{[k_n,\ell_n]}(x_n)\right)$$

be a specification in (X, F). We say that S is *an N-spaced specification* if, for each $j \in \{1, 2, \ldots, n-1\}$,

$$k_{j+1} - \ell_j \geq N.$$

Definition

Let (X, F) be a CR-dynamical system, let d be the metric on X, let N be a positive integer, let $\epsilon > 0$, let $y \in X$ and let

$$S = \left(F^{[k_1,\ell_1]}(x_1), F^{[k_2,\ell_2]}(x_2), F^{[k_3,\ell_3]}(x_3), \ldots, F^{[k_n,\ell_n]}(x_n)\right)$$

be an N-spaced specification in (X, F). We say that S is *ϵ-traced in (X, F) by y* if, for each $i \in \{1, 2, \ldots, n\}$ and for each $j \in \{k_i, k_i + 1, k_1 + 2 \ldots, \ell_i\}$,

$$d\left(F^j(y), F^j(x_i)\right) \leq \epsilon.$$

Definition

Let (X, F) be a CR-dynamical system. We say that (X, F) *has specification property* if for every $\epsilon > 0$ there is a positive integer N such that for any N-spaced specification S in (X, F) there exists $y \in X$ such that S is ϵ-traced in (X, F) by y.

Theorem (BEJK, 2024.)

Let (X, F) be a CR-dynamical system. If there exists a positive integer n_0 such that $F^{n_0}(x) \cap F^{n_0}(y) \neq \emptyset$, for all $x, y \in X$, then (X, F) has specification property.

Initial SP in CR-dynamical systems

- Let (X, F) be a CR-dynamical system, let $x \in X$ and let ℓ be a non-negative integer. We say that $F^{[0,\ell]}(x)$ ia *an initial ℓ-orbit segment of the point x*.

Definition

Let (X, F) be a CR-dynamical system, let n be a positive integer and, for each $j \in \{1, 2, \dots, n\}$, let

- ℓ_j be non-negative integer
- $x_j \in X$.

We say that the n-tuple (of sets)

$$\left(F^{[0,\ell_1]}(x_1), F^{[0,\ell_2]}(x_2), F^{[0,\ell_3]}(x_3), \dots, F^{[0,\ell_n]}(x_n)\right)$$

is *an initial n-specification* (or just *an initial specification*) in (X, F).

Definition

Let (X, F) be a CR-dynamical system, let d be the metric on X, let n and $m_1, m_2, \ldots, m_{n-1}$ be positive integers, let $\epsilon > 0$, let $y \in X$ and let

$$S = \left(F^{[0,\ell_1]}(x_1), F^{[0,\ell_2]}(x_2), F^{[0,\ell_3]}(x_3), \ldots, F^{[0,\ell_n]}(x_n)\right)$$

be an initial specification in (X, F). We say that S is $(\epsilon, m_1, m_2, \ldots, m_{n-1})$*-traced in* (X, F) *by* y if, for each $i \in \{1, 2, \ldots, n\}$ and for each $j \in \{0, 1, 2 \ldots, \ell_i\}$,

$$d\left(F^{\ell_1+m_1+\ell_2+m_2+\cdots+\ell_{i-1}+m_{i-1}+j}(y), F^j(x_i)\right) \le \epsilon.$$

Definition

Let (X, F) be a CR-dynamical system. We say that (X, F) *has initial specification property* if for every $\epsilon > 0$ there is a positive integer N such that

- for each positive integer n
- for all positive integers $m_1, m_2, \ldots, m_{n-1}$ such that, for each $i \in \{1, 2, \ldots, n-1\}$, $m_i \geq N$

and for any initial specification S in (X, F) there exists $y \in X$ such that S is $(\epsilon, m_1, m_2, \ldots, m_{n-1})$-traced in (X, F) by y.

SP vs initial SP in CR-dynamical systems

The following example shows that in general, the specification property is not equivalent to the initial specification property.

- Let $X = [0, 1]$ and let $F = [0, 1] \times \{1\}$. Note that (X, F) has specification property but it does not have initial specification property.

However, the following theorem holds.

Theorem (BEJK, 2024.)

Let (X, F) be a CR-dynamical system. If (X, F) has initial specification property, then (X, F) has specification property.

The following example shows that even if $p_1(F) = p_2(F) = X$, the specification property and initial specification property may not be equivalent.

Example

Let $X = [0, 1]$ and let

$$F = \left(\left[0, \frac{1}{2}\right] \times \{0\}\right) \cup \left(\left[\frac{1}{2}, 1\right] \times \{1\}\right) \cup (\{1\} \times [0, 1]) .$$

Note that (X, F) has specification property but it does not have initial specification property.

Theorem (BEJK, 2024.)

Let (X, F) be a CR-dynamical system. If there is a positive integer n_0 such that $F^{n_0}(x) = X$, for each $x \in X$, then (X, F) has specification property as well as initial specification property.

Theorem (BEJK, 2024.)

Let (X, F) be a CR-dynamical system. If there is a positive integer n_0 such that $F^{n_0}(x) = X$, for each $x \in X$, then (X, F) has specification property as well as initial specification property.

Example

Let $X = [0, 1]$ and let

$$F = \left(\left[0, \frac{1}{2}\right] \times \{0\}\right) \cup \left(\{0\} \times \left[0, \frac{1}{2}\right]\right) \cup \left(\left[\frac{1}{2}, 1\right] \times \{1\}\right) \cup \left(\{1\} \times \left[\frac{1}{2}, 1\right]\right).$$

Note that, for each $x \in X$, $F^4(x) = X$. Therefore, by the previous theorem, (X, F) has specification property as well as initial specification property.

References

[1] D. Kwietniak, M. Lacka and P. Oprocha, *A panorama of specification-like properties and their consequences,* arXiv:1503.07355v2 [math.DS], 2015.

Thank you for your attention!