The Baire property and precompact duality

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This is joint work with Professors María Vicenta Ferrer (*Universitat Jaume I*), Salvador Hernández (*Universitat Jaume I*), and F. Javier Trigos-Arrieta (*California State University*).

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Groups $\mathbb{T}$ and $\widehat{G}$

All groups considered in this presentation are assumed to be **Abelian or commutative.**

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$$\widehat{G} := \{\phi : G \to \mathbb{T} : \phi \text{ continuous homomorphism}\}.$$

It becomes a group declaring

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Finite-open topology and totally bounded groups

3. A topological group G is **MAP**, if whenever $g \in G \setminus \{0\}$, there is $\phi \in \widehat{G}$ with $\phi(g) \neq 0$.

4. If $F \subseteq G$, and $V \subseteq \mathbb{T}$, let $(F, V) := \{\phi \in \widehat{G} : \phi[F] \subseteq V\}$. Consider

$$\mathcal{B} = \{(F, V) : F \text{ finite in } G,\ V \text{ open in } \mathbb{T}\}.$$

$\mathcal{B}$ is a subbase of the group toplogy τ_p on $\widehat{G}$ called the ***finite-open*** *topology* on $\widehat{G}$. We define $\widehat{G}_p := (\widehat{G}, \tau_p)$.

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$$G \text{ is totally bounded} \iff \widehat{(\widehat{G}_p)}_p \cong G$$

[Comfort-Ross, Raczkowski, Trigos-Arrieta]

7. TFAE for a topological group G.
   a. No non-empty open set U of G is of *first category*, *i. e.*, U cannot be written as a *countable* union of sets whose closure has empty interior.
   b. G is not of *first category*, *i. e.*, G cannot be written as a *countable* union of sets whose closure has empty interior.
   c. The intersection of *countable* many dense open sets of G is dense in G.

If G satisfies either one of the properties listed, we say that G is a ***Baire*** group or that G is of ***second category*** (in general these two concepts are different for spaces that are not topological groups).

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Some history

A. In 1955, Leptin proved that $\widehat{G}_p$, with G **compact**, hence Baire, does not contain infinite compact sets.

B. Let G be a **totally bounded Baire group**. In 2012, Bruguera and Tkachenko (=[BT]) proved (a) that $\widehat{G}_p$ does not contain non-trivial convergent sequences, and (b) if G satisfies ***the open refinement condition***, then $\widehat{G}_p$ does not contain infinite compact sets.

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C. In 2017, Chasco, Domínguez and Tkachenko (=[CDT]) proved that $\widehat{G}_p$, with G a **totally bounded**, ***bounded torsion***, **Baire group**, does not contain infinite compact sets. Then, they ask 'Can there be infinite compacts in the dual space of a Baire and totally bounded group?'.

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A compact infinite space K that is not scattered contains a closed subspace L such that L has a countable cover $\{V_n\}$ of open sets, each of cardinality $\geq \mathfrak{c}$, such that for every open set V of L there is n_0 with $V_{n_0} \subseteq V$.

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Some algebra

Definition 2

A non-empty subset A of a group G is said to be **independent**, *if whenever $x_1, \dots, x_n \in A$ and $k_1, \dots, k_n \in \mathbb{Z}$, then* $k_1x_1 + \cdots + k_nx_n = 0 \implies k_1x_1 = \cdots = k_nx_n = 0$.

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Main Theorem

Theorem 1

Let G be a totally bounded abelian Baire group. Then every compact subset of $\widehat{G}_p$ is finite.

Proof of the Main Theorem

Let K be an infinite compact set in the dual group and assume that K is not scattered WLOG.

- $I_{2m} := (-1/8, 1/8)$, $I_{2m+1} := (-1/2, -1/4) \cup (1/4, 1/2]$.

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- By Lemma 2, there is $\emptyset \neq L \subseteq K$ closed and an open countable cover $\{V_n\}$ such that every open subset in L contains an element of the open cover. Define

$$S_n = \{f \in \widehat{G}_p : \exists \{x_{nk}\}_{k<\omega} \subseteq V_n, \text{ with } f(x_{n,k}) \in I_k \; \forall k < \omega\}.$$

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- $\bigcap_{n=1}^{\infty} S_n$ contains a dense G_δ subset of $\widehat{G}_p \Rightarrow \exists\, \phi \in \bigcap_{n=1}^{\infty} S_n$.
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$$1/8 < |\phi(x_{n_0,0}) - \phi(x_{n_0,1})| \leq |\phi(x_{n_0,0}) - \phi(w)| + |\phi(w) - \phi(x_{n_0,1})| < 1/8\#.$$

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A new class of g-barrelled groups

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g-dense groups

Definition 3 (Def. 5.4.1 in [AD])

Let K be a topological group and $u = (u_n)$ be a sequence in $\widehat{K}$. Let

$$s_u(K) := \{x \in K : \lim_n u_n(x) = 0 \text{ in } \mathbb{T}\}.$$

A subgroup G of K is called:

- ***characterizable**, if there exists $u = (u_n)$ in $\widehat{K}$ such that $G = s_u(K)$;*
- ***g-closed**, if G is the intersection of characterizable subgroups of K;*
- ***g-dense**, if $G \subseteq s_u(K)$ for some $u = (u_n)$ in $\widehat{K}$ yields $s_u(K) = K$.*

g-dense doesn't imply g-barrelled

Aussenhofer and Dikranjan ask [AD] (Question 11.3.3) whether every totally bounded group G that is g-dense in its completion must be a g-barrelled group.

Corollary 1

Set $\omega^* := \beta\omega \setminus \omega$. The group $C_p(\omega^*, \mathbb{T})$ is g-dense in its completion but is not g-barrelled.

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More results involving C_p-theory

The following is a collection of results that follow from the main Theorem. We mention them without providing too much detail.

1. If X is a μ-space containing an infinite compact subspace, then $C_p(X, \mathbb{T})$ is not Baire. In particular, this is the case if X is compact and infinite.

2. If K is a compact zero-dimensional space that is not scattered, then $C_p(K, \{0, 1\})$, is not Baire. (Roman Pol originally communicated this and its proof in a private email.)

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Reflexivity

Definition 4

A topological group is said to be **reflexive** *if the evaluation map $e : G \to \widehat{(\widehat{G}_k)}_k$ is a topological isomorphism, where the duals are equipped with the compact-open topology.*

This is the topology used in the Pontryagin-van Kampen Theorem for LCAGs.

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2. Totally bounded Baire groups without infinite compact subsets are reflexive if satisfying the Open Refinement Condition [BT].

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Reflexivity

The following extends these results.

Corollary 2

A totally bounded Baire group G without infinite compact subsets is reflexive.

Proof: By the Comfort-Ross Theorem, G has the weak topology generated by $\widehat{G}$, *i. e.*, G is topologically isomorphic to $\left(\widehat{\widehat{G}_p}\right)_p$. Furthermore, since G contains no infinite compact subsets, the compact-open topology on the dual group $\widehat{G}$ coincides with the finite-open topology. Hence, it follows that G is Pontryagin reflexive.

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For full proofs and references, the reader is invited to see the preprint in ArXiV. Thank you very much for your attention.

The Baire property and precompact duality

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