Persistent Properties in Uniform Box Products

Jocelyn Bell

Hobart and William Smith Colleges

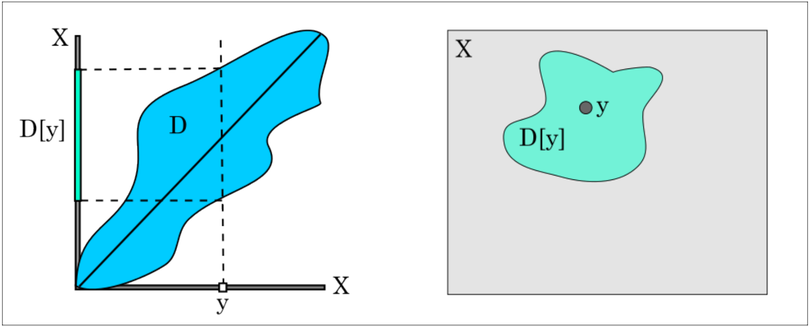
All spaces are Hausdorff

All spaces are Hausdorff

Most will also be uniform

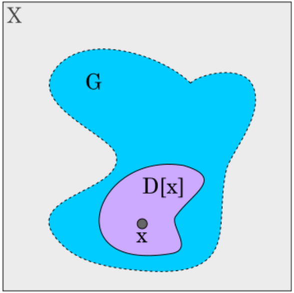
A Diagonal Uniformity on a Set X

A collection $\mathbb{D}$ of entourages of the diagonal of $X \times X$



The Uniform Topology on X

A set G is open if for all $x \in G$ there is $D \in \mathbb{D}$ with $D[x] \subseteq G$



The Uniformity of Uniform Convergence

A uniformity on a space of functions

Definition

(Bourbaki, 1949). Suppose $(X, \mathbb{D})$ is a uniform space, Y is a topological space, and X^Y is the set of all continuous functions $Y \to X$

Definition

(Bourbaki, 1949). Suppose $(X, \mathbb{D})$ is a uniform space, Y is a topological space, and X^Y is the set of all continuous functions $Y \to X$

For every $D \in \mathbb{D}$, define an entourage $\overline{D}$ of the diagonal of $X^Y \times X^Y$ by

$$\overline{D} = \{(f, g) : (f(x), g(x)) \in D \text{ for all } x \in Y\}$$

Definition

(Bourbaki, 1949). Suppose $(X, \mathbb{D})$ is a uniform space, Y is a topological space, and X^Y is the set of all continuous functions $Y \to X$

For every $D \in \mathbb{D}$, define an entourage $\overline{D}$ of the diagonal of $X^Y \times X^Y$ by

$$\overline{D} = \{(f, g) : (f(x), g(x)) \in D \text{ for all } x \in Y\}$$

Then $\left\{\overline{D} : D \in \mathbb{D}\right\}$ is a uniformity on the set X^Y called the **uniformity of uniform convergence**

Uniform Box Product

A specific uniformity of uniform convergence

Definition

(Williams, 2001) Suppose $(X, \mathbb{D})$ is a uniform space. The set $X^{\mathbb{N}}$ with the uniform topology generated by the uniformity of uniform convergence is called the **uniform box product** of $(X, \mathbb{D})$. We will use the notation

$$(X, \mathbb{D})^{\mathbb{N}}$$

Uniform Box Product

A specific uniformity of uniform convergence

Definition

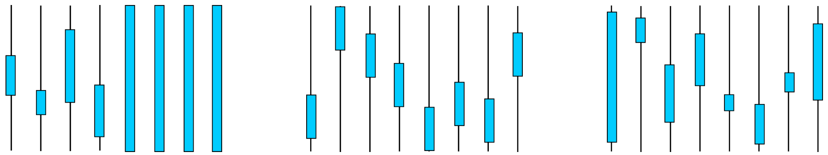
(Williams, 2001) Suppose $(X, \mathbb{D})$ is a uniform space. The set $X^{\mathbb{N}}$ with the uniform topology generated by the uniformity of uniform convergence is called the **uniform box product** of $(X, \mathbb{D})$. We will use the notation

$$(X, \mathbb{D})^{\mathbb{N}}$$

Why the name?

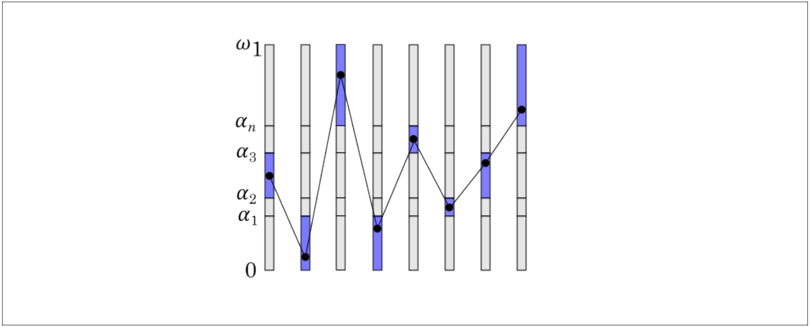
3 Topologies on $X^{\mathbb{N}}$

X is a uniform space



Uniform Box Product of $[0, \omega_1]$

A neighborhood of a point in this product



Proceed with Caution

The choice of uniformity matters!

Consider the discrete space $\mathbb{N}$

Proceed with Caution

The choice of uniformity matters!

Consider the discrete space $\mathbb{N}$

1. Let $\mathbb{D}_1$ be the uniformity of all subsets of $\mathbb{N} \times \mathbb{N}$
    - So the diagonal $\triangle$ is in $\mathbb{D}_1$

Proceed with Caution

The choice of uniformity matters!

Consider the discrete space $\mathbb{N}$

1. Let $\mathbb{D}_1$ be the uniformity of all subsets of $\mathbb{N} \times \mathbb{N}$
   - So the diagonal $\triangle$ is in $\mathbb{D}_1$
2. Let $\mathbb{D}_2$ be the uniformity inherited from the one-point compactification of $\mathbb{N}$
   - A base for this uniformity is squares of finite open covers of $\mathbb{N} \cup \{\infty\}$ restricted to $\mathbb{N} \times \mathbb{N}$

Proceed with Caution

The choice of uniformity matters!

Consider the discrete space $\mathbb{N}$

1. Let $\mathbb{D}_1$ be the uniformity of all subsets of $\mathbb{N} \times \mathbb{N}$
   - So the diagonal $\triangle$ is in $\mathbb{D}_1$
2. Let $\mathbb{D}_2$ be the uniformity inherited from the one-point compactification of $\mathbb{N}$
   - A base for this uniformity is squares of finite open covers of $\mathbb{N} \cup \{\infty\}$ restricted to $\mathbb{N} \times \mathbb{N}$

Then $(\mathbb{N}, \mathbb{D}_1)^{\mathbb{N}}$ is discrete, while $(\mathbb{N}, \mathbb{D}_2)^{\mathbb{N}}$ is not.

Persistent Properties in Products

Which properties persist in each of the three products?

If X is a space with a certain property, does $X^{\mathbb{N}}$ also have the property?

Persistent Properties in Products

Which properties persist in each of the three products?

If X is a space with a certain property, does $X^{\mathbb{N}}$ also have the property?

- For example, finite powers of a normal space need not be normal
  - Nor even pseudonormal

Persistent Properties in Products

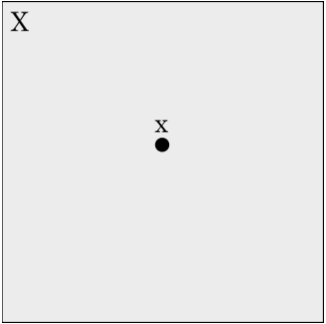
Which properties persist in each of the three products?

If X is a space with a certain property, does $X^{\mathbb{N}}$ also have the property?

- For example, finite powers of a normal space need not be normal
  - Nor even pseudonormal
- But some properties are preserved . . .

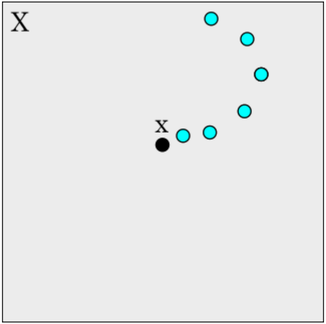
Arhangel'skii's α_2 property

Fix a point x in a space X



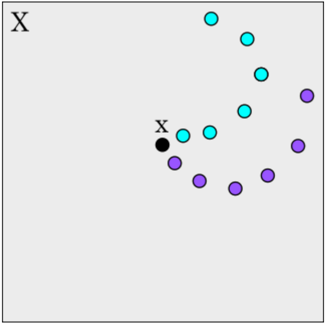
Arhangel'skii's α_2 property

And a countable collection of sequences all converging to x



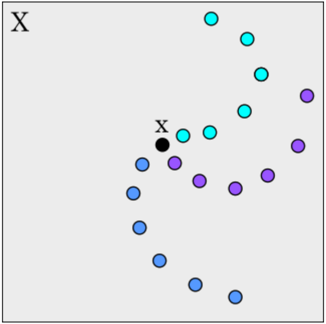
Arhangel'skii's α_2 property

And a countable collection of sequences all converging to x



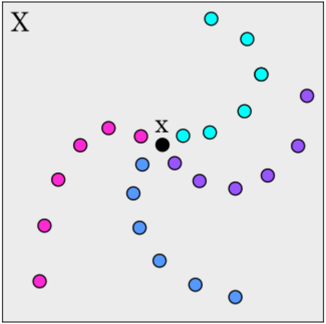
Arhangel'skii's α_2 property

And a countable collection of sequences all converging to x



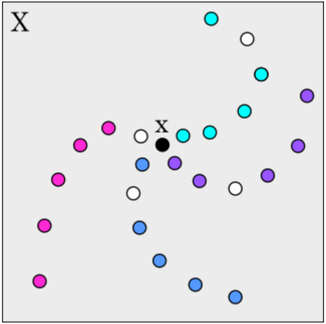
Arhangel'skii's α_2 property

And a countable collection of sequences all converging to x



Arhangel'skii's α_2 property

Can you always find a "diagonal" sequence also converging to x?



α_2 points and α_2 spaces

Definition

(Arhangel'skii, 1972).[1] Let X be a space and $x \in X$. The point x is an α_2-**point** if for every countable collection of sequences $S_1, S_2, \ldots$ converging to x,

[1] Initially called α_5, but $\alpha_5 = \alpha_2$ was shown by Nogura in 1985

α_2 points and α_2 spaces

Definition

(Arhangel'skii, 1972).[1] Let X be a space and $x \in X$. The point x is an α_2-**point** if for every countable collection of sequences $S_1, S_2, \ldots$ converging to x,
there is a sequence A converging to x so that for each n,

$$\mathrm{ran} A \cap \mathrm{ran} S_n \neq \emptyset$$

[1] Initially called α_5, but $\alpha_5 = \alpha_2$ was shown by Nogura in 1985

α_2 points and α_2 spaces

Definition

(Arhangel'skii, 1972).[1] Let X be a space and $x \in X$. The point x is an α_2-**point** if for every countable collection of sequences $S_1, S_2, \ldots$ converging to x,
there is a sequence A converging to x so that for each n,

$$\operatorname{ran} A \cap \operatorname{ran} S_n \neq \emptyset$$

If every point of X is an α_2-point, then X is an α_2-**space.**

[1] Initially called α_5, but $\alpha_5 = \alpha_2$ was shown by Nogura in 1985

Products of α_2 spaces

This property is persistent

Theorem

(Nogura, 1985). A product of countably many α_2 spaces is an α_2 space.

Products of α_2 spaces

This property is persistent

Theorem

(Nogura, 1985). A product of countably many α_2 spaces is an α_2 space.

But the proof does not extend to our other products because . . .

Convergence!

Different products, different conditions for convergence

- Product topology: point-wise convergence
- Uniform box topology: uniform convergence
- Box topology: “uniformly equivalent” convergence

Box products of α_2 spaces

Does the property persist?

Box products of α_2 spaces

Does the property persist?

No! $[0, \omega]^{\mathbb{N}}$ with the box topology is not α_2

Box products of α_2 spaces

Does the property persist?

No! $[0, \omega]^{\mathbb{N}}$ with the box topology is not α_2

- For each n, consider the sequence $S_n = \langle a_1^n, a_2^n, \ldots \rangle$ converging to $(\omega, \omega, \ldots)$ defined by

$$a_i^n(j) = \begin{cases} i & \text{if } j \leq n \\ \omega & \text{if } j > n \end{cases}$$

Box products of α_2 spaces

Does the property persist?

No! $[0,\omega]^{\mathbb{N}}$ with the box topology is not α_2

- For each n, consider the sequence $S_n = \langle a_1^n, a_2^n, \ldots \rangle$ converging to $(\omega, \omega, \ldots)$ defined by

$$a_i^n(j) = \begin{cases} i & \text{if } j \leq n \\ \omega & \text{if } j > n \end{cases}$$

- A box product of infinitely many spaces that each have non-trivial convergent sequences cannot be α_2

Uniform box products of α_2 spaces

Does the property persist?

Uniform box products of α_2 spaces

Does the property persist?

I don't know. But:

Uniform box products of α_2 spaces

Does the property persist?

I don't know. But:

Theorem

If X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from the one-point compactification of X, then $(X, \mathbb{D})^{\mathbb{N}}$ is an α_2 space.

Uniform box products of α_2 spaces

Does the property persist?

I don't know. But:

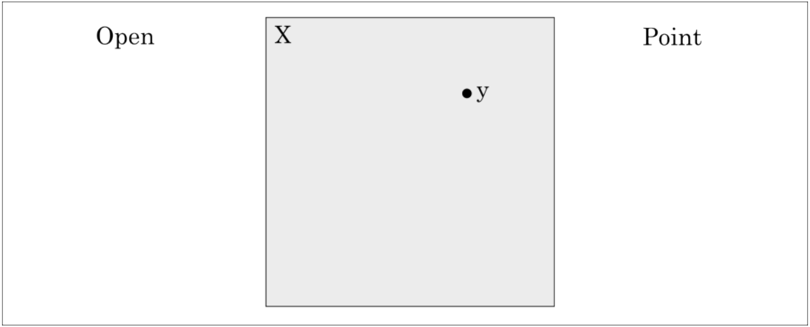
Theorem

If X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from the one-point compactification of X, then $(X, \mathbb{D})^{\mathbb{N}}$ is an α_2 space.

The proof involves a topological game, related to and inspired by Gruenhage's W-game.

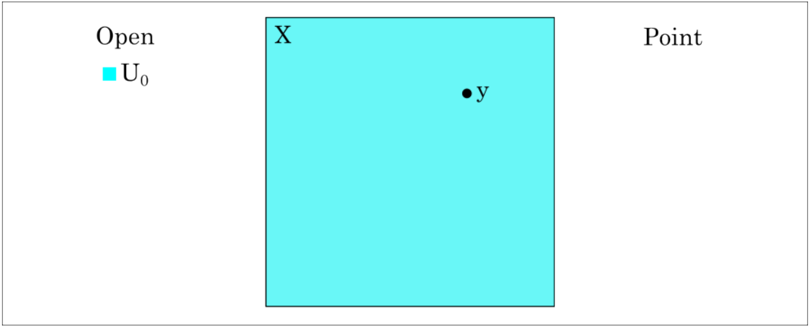
Gruenhage's W-game

Fix a point y in a topological space X



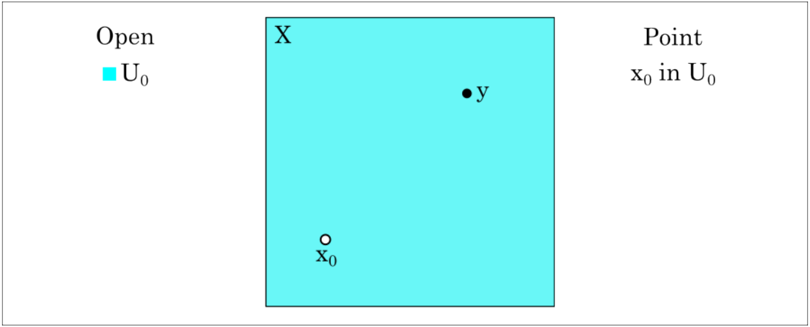
Gruenhage's W-game

Open chooses an open set U_0 (usually the whole space initially) containing y



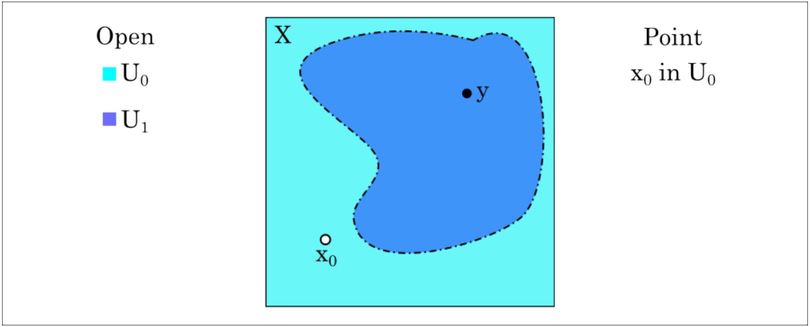
Gruenhage's W-game

Point chooses any $x_0 \in U_0$



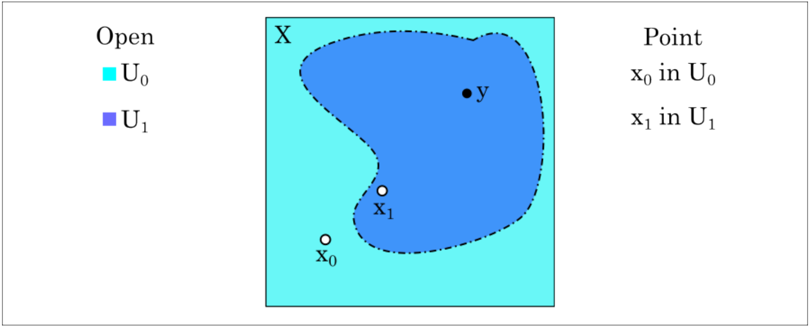
Gruenhage's W-game

Open chooses an open set U_1 containing y



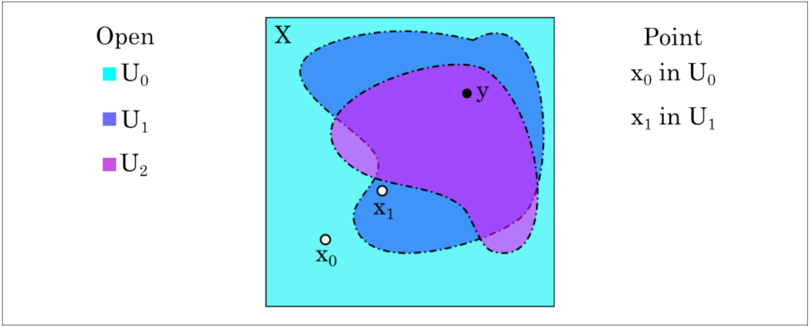
Gruenhage's W-game

Point responds with any $x_1 \in U_1$



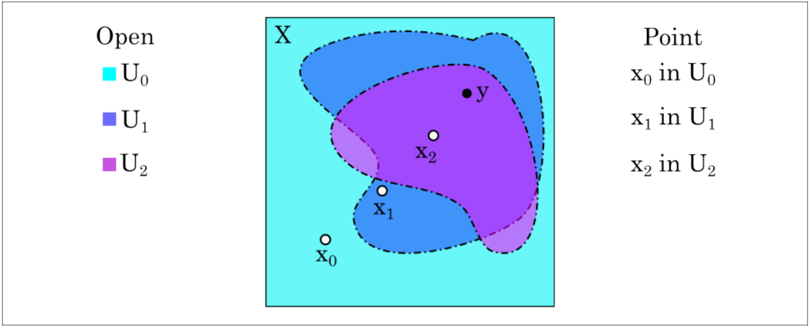
Gruenhage's W-game

Open chooses open U_2 containing y



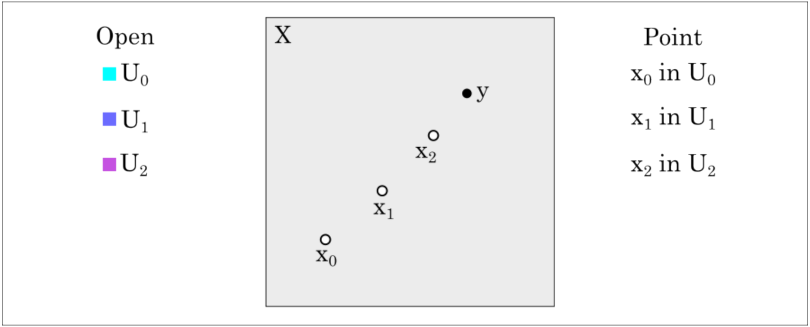
Gruenhage's W-game

Point chooses any $x_2 \in U_2$ and the game continues



Winning Gruenhage's W-game

Open wins the game if Point's choices converge to y



W-spaces

W for “win”

Definition

(Gruenhage) A point $x \in X$ is a W-point if the first player has a winning strategy in the W-game at x. A space X is a W-space if every point of X is a W-point.

W-spaces

W for "win"

Definition

(Gruenhage) A point $x \in X$ is a W-point if the first player has a winning strategy in the W-game at x. A space X is a W-space if every point of X is a W-point.

Theorem

(Gruenhage) A product of countably many W-spaces is a W-space, and every W-space is an α_2-space.

Proximal Game

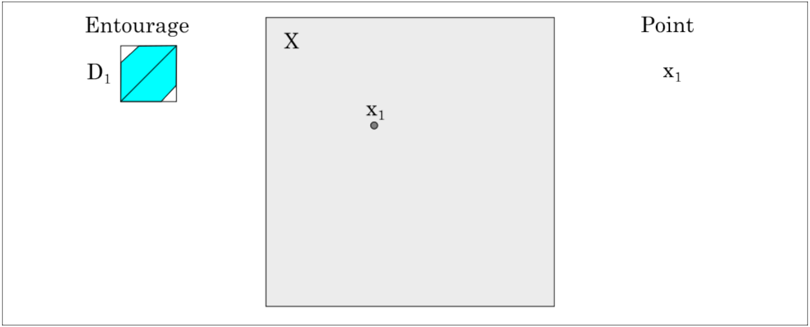
Suppose $(X, \mathbb{D})$ is a uniform space

Proximal Game

Entourage moves first and chooses any element of $\mathbb{D}$

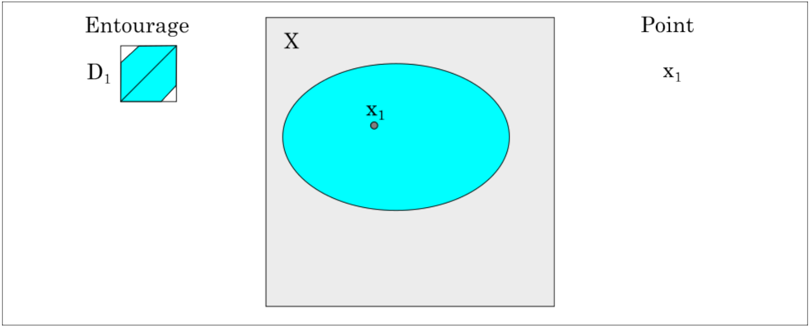
Proximal Game

Point responds with any $x_1 \in X$ . . .



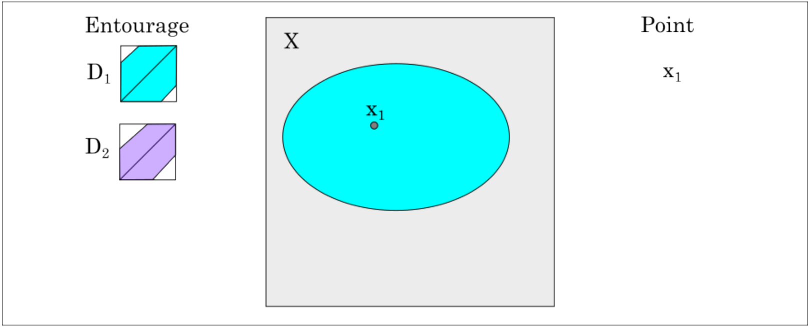
Proximal Game

. . . knowing he will be stuck choosing from $D_1[x_1]$ in the next round



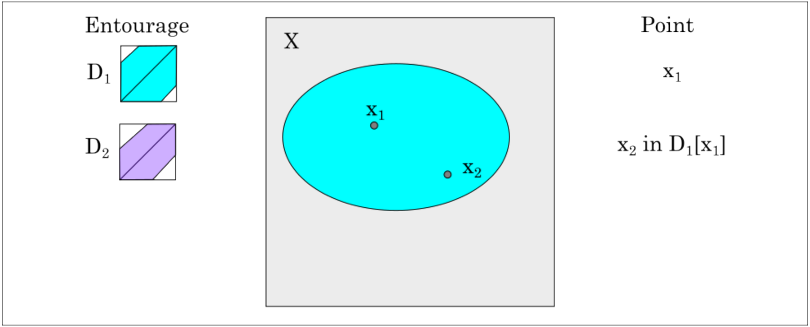
Proximal Game

Entourage chooses any $D_2 \in \mathbb{D}$



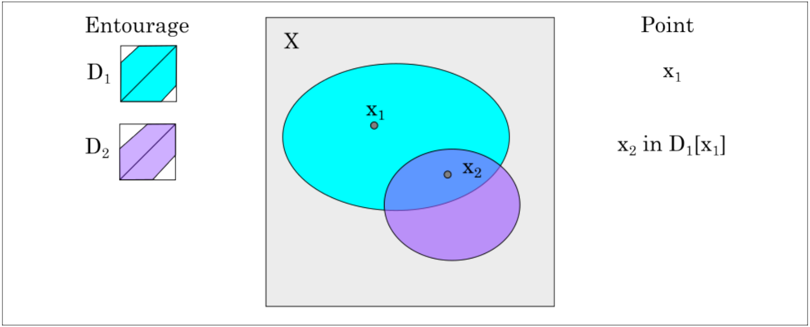
Proximal Game

And Point chooses any $x_2 \in D_1[x_1]$ . . .



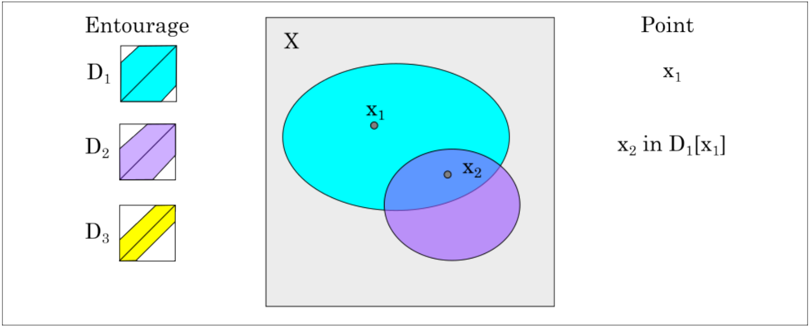
Proximal Game

. . . knowing he will be stuck in $D_2[x_2]$ in the next round



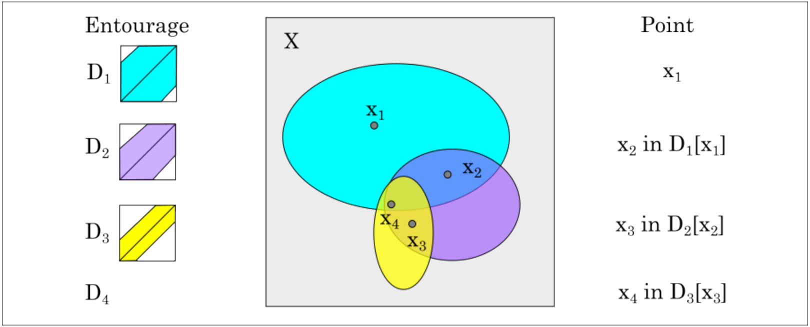
Proximal Game

Entourage chooses any $D_3 \in \mathbb{D}$



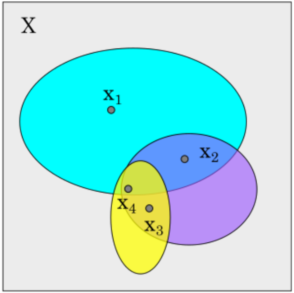
Proximal Game

. . . and the game continues



Winning the Proximal Game

Entourage wins the game if Point's choices form a convergent sequence



Definition
(B.) A uniform space $(X, \mathbb{D})$ is **proximal** if Entourage has a winning strategy in the proximal game.

Proximal Spaces

Definition and some properties

Definition

(B.) A uniform space $(X, \mathbb{D})$ is **proximal** if Entourage has a winning strategy in the proximal game.

- Proximal spaces are W-spaces, collectionwise normal, countably paracompact, and a product of countably many proximal spaces is proximal (B.)

Definition

(B.) A uniform space $(X, \mathbb{D})$ is **proximal** if Entourage has a winning strategy in the proximal game.

- Proximal spaces are W-spaces, collectionwise normal, countably paracompact, and a product of countably many proximal spaces is proximal (B.)
- A compact space is proximal if and only if it is Corson compact (Clontz & Gruenhage)

Definition

(B.) A uniform space $(X, \mathbb{D})$ is **proximal** if Entourage has a winning strategy in the proximal game.

- Proximal spaces are W-spaces, collectionwise normal, countably paracompact, and a product of countably many proximal spaces is proximal (B.)
- A compact space is proximal if and only if it is Corson compact (Clontz & Gruenhage)
- The uniform box product of a compact proximal space is proximal (Hernandez-Gutierrez & Szeptycki)

Definition

(B.) A uniform space $(X, \mathbb{D})$ is **proximal** if Entourage has a winning strategy in the proximal game.

- Proximal spaces are W-spaces, collectionwise normal, countably paracompact, and a product of countably many proximal spaces is proximal (B.)
- A compact space is proximal if and only if it is Corson compact (Clontz & Gruenhage)
- The uniform box product of a compact proximal space is proximal (Hernandez-Gutierrez & Szeptycki)
- But $[0, \omega_1]$ is not proximal . . . so we'll need something else

ω-Proximal Game

A modification of the proximal game

Definition

(B.) Suppose $(X, \mathbb{D})$ is a uniform space and $S \subseteq X$. X is S-**proximal** provided Entourage has a winning strategy in the proximal game, where Point is subject to the additional condition that his choices must be elements of S.

ω-Proximal Game

A modification of the proximal game

Definition

(B.) Suppose $(X, \mathbb{D})$ is a uniform space and $S \subseteq X$. X is S-**proximal** provided Entourage has a winning strategy in the proximal game, where Point is subject to the additional condition that his choices must be elements of S.

A uniform space $(X, \mathbb{D})$ is ω-**proximal** if X is S-proximal for every countable subset S of X.

ω-Proximal Game

A modification of the proximal game

Definition

(B.) Suppose $(X, \mathbb{D})$ is a uniform space and $S \subseteq X$. X is S-**proximal** provided Entourage has a winning strategy in the proximal game, where Point is subject to the additional condition that his choices must be elements of S.

A uniform space $(X, \mathbb{D})$ is ω-**proximal** if X is S-proximal for every countable subset S of X.

An ω-proximal space is pseudonormal and α_2, and countable products of ω-proximal spaces are ω-proximal (B.)

Definition

(B.) Suppose $(X, \mathbb{D})$ is a uniform space and $S \subseteq X$. X is S-**proximal** provided Entourage has a winning strategy in the proximal game, where Point is subject to the additional condition that his choices must be elements of S.

A uniform space $(X, \mathbb{D})$ is ω-**proximal** if X is S-proximal for every countable subset S of X.

An ω-proximal space is pseudonormal and α_2, and countable products of ω-proximal spaces are ω-proximal (B.)

$[0, \omega_1]$ is ω-proximal but not proximal, and . . .

Uniform Box Products of Ordinals

And (very slightly) beyond ordinal spaces

Theorem

(B.) Suppose X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from its one-point compactification. Then $(X, \mathbb{D})^{\mathbb{N}}$ is ω-proximal (and so is both pseudonormal and α_2).

Theorem

(B.) Suppose X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from its one-point compactification. Then $(X, \mathbb{D})^{\mathbb{N}}$ is ω-proximal (and so is both pseudonormal and α_2).

The properties I actually needed were:

1. local compactness
   - ensures the space has a Hausdorff one-point compactification

Uniform Box Products of Ordinals

And (very slightly) beyond ordinal spaces

Theorem

(B.) Suppose X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from its one-point compactification. Then $(X, \mathbb{D})^{\mathbb{N}}$ is ω-proximal (and so is both pseudonormal and α_2).

The properties I actually needed were:

1. local compactness
   - ensures the space has a Hausdorff one-point compactification
2. σ-compactness

Uniform Box Products of Ordinals

And (very slightly) beyond ordinal spaces

Theorem

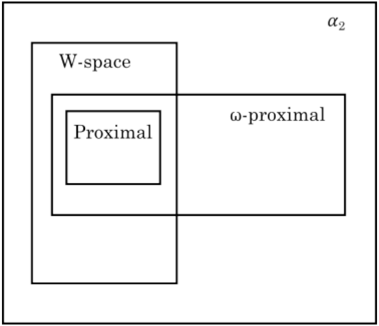
(B.) Suppose X is an ordinal space and $\mathbb{D}$ is the uniformity on X inherited from its one-point compactification. Then $(X, \mathbb{D})^{\mathbb{N}}$ is ω-proximal (and so is both pseudonormal and α_2).

The properties I actually needed were:

1. local compactness
   - ensures the space has a Hausdorff one-point compactification
2. σ-compactness
3. the closure of every countable set is countable
   - this implies the space must also be scattered and zero-dimensional

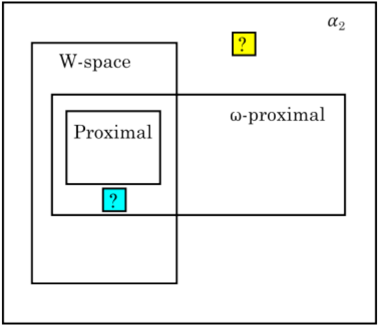
Relationships Among the Properties

For uniform spaces



Relationships Among the Properties

For uniform spaces



A Summary of Persistent Properties

Which properties persist in which products if X is a uniform space?

	Tychonoff	Box	Uniform Box	$+\ X$ compact
α_2	Yes	No	?	?
ω-proximal	Yes	No	?	?
proximal	Yes	No	No	Yes
W-space	Yes	No	No	?

Obrigada!

Thanks for listening!