Ergodic theory for operators and applications to generalized Cesàro operators

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Joint work with A.A. Albanese and W.J. Ricker

Aim of the lecture

AIM

In the first part we review classical results about power bounded and mean ergodic operators acting on Banach and more general spaces. They will be utilized to investigate the behaviour of generalized Cesàro operators when acting on sequence spaces and spaces of analytic functions on the disc of the complex plane.

In the second part we report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Aim of the lecture

Ergodic theory has its origins in statistical mechanics, which is a mathematical framework that applies probability theory to large assemblies of microscopic entities.

Theorem. Birkhoff's ergodic theorem. 1931

Let (X, Σ, μ) a measure space, $\mu(X) = 1$, and let $\varphi : X \to X$ satisfy $\mu(\varphi^{-1}(E)) = \mu(E)$ for each $E \in \Sigma$. For each $f \in L_1(X, \mu)$ the sequence of averages

$$A_N f := \frac{1}{N} \sum_{n=0}^{N-1} f(\varphi^n(x)), \quad N \in \mathbb{N},$$

converges μ-almost everywhere $x \in X$. Moreover, if $\overline{f}$ is the function defined by the limit, then $\overline{f} \in L_1(\Omega, \mu)$, is φ-invariant and $A_N f \longrightarrow \overline{f}$ in $L_1(\Omega, \mu)$ as $N \to \infty$.

This theorem contains von Neumann L^2 ergodic theorem, that was known to Birkhoff and motivated his work.

First Part

Ergodic Theory of linear operators on locally convex spaces. Power bounded and (uniformly) mean ergodic operators

Averages of complex numbers

Let $u \in \mathbb{C}$ be a complex number such that $|u| = 1$. Define

$$v_n := \frac{1}{n}\sum_{j=1}^{n} u^j.$$

If $u = 1$, then $v_n = 1$ for each $n \in \mathbb{N}$.

If $u \neq 1$, then

$$|v_n| = \left|\frac{u - u^{n+1}}{n(1-u)}\right| \leq \frac{2}{n|1-u|} \to 0, \quad n \to \infty.$$

Therefore the means $(v_n)_n$ converge. This helps us to motivate the following results.

Von Neumann' Theorem (1932)

Theorem (von Neumann (1932), Riesz-Nagy (1943))

Let T be a contraction on a Hilbert space H, that is $\|T\| \leq 1$. Then, for each $f \in H$, the following limit exists

$$P_T(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j(f).$$

Moreover $H = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(H)}$ is an orthogonal decomposition and P_T is the orthogonal projection onto the set $\mathrm{Ker}(I - T)$ of fixed points of T.

This theorem was extended by F: Riesz (1938) for operators $T : L^p(0.1) \to L^p(0,1),\ 1 < p < \infty$ with $\|T\| \leq 1$ and by Lorch (1938) for operators $T : X \to X$ on a reflexive Banach space with $\sup_n \|T^n\| < \infty$.

Context.

X **is a Hausdorff locally convex space (lcs) over the field of complex numbers** $\mathbb{C}$**.**

$\mathcal{L}(X)$ **(resp.** $\mathcal{K}(X)$**) is the space of all continuous (resp. compact) linear operators** $T : X \to X$ **on** X**.**

$T \in \mathcal{L}(X)$ **is compact if there is a neighbourhood** U **of** 0 **such that** $T(U)$ **is relatively compact on** X**.**

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a barrelled space, for example if X is a complete metrizable locally convex spaces, then an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1}$$

exist in X.

Cesàro bounded operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$, then T is said to be Cesàro bounded.

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ converges uniformly on the bounded subsets of X to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

We denote by $\mathcal{L}_b(X)$ the space of operators $\mathcal{L}(X)$ endowed with the topology of uniform convergence on the bounded subsets of X.

If X is a Banach space, $\mathcal{L}_b(X)$ is the space of operators $\mathcal{L}(X)$ endowed with the operator norm.

Mean ergodic properties. Definitions

$\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, for $n \geq 2$

$\lim_{n\to\infty} \frac{T^n}{n} x = 0$ for each $x \in X$ whenever T is mean ergodic. Moreover, if T is uniformly mean ergodic, then $\lim_{n\to\infty} \frac{T^n}{n} = 0$ uniformly on the bounded sets.

Example

The linear map $T : \mathbb{C}^2 \to \mathbb{C}^2$, $T(z_1, z_2) := (-z_1 + 2z_2, -z_2)$, is Cesáro bounded but not mean ergodic.

Mean ergodic properties. Main Theorems

Theorem. Eberlein.

Let $T \in \mathcal{L}(X)$ be Cesàro bounded. The operator T is mean ergodic if and only if the following two conditions are satisfied.

(1) $\lim_{n\to\infty} \frac{T^n}{n}x = 0, \ x \in X$.

(2) $(T_n x)_n$ is $\sigma(X, X')$-relatively compact for every $x \in X$.

Mean ergodic properties. Main Theorems

Theorem. Yosida.

Let $T \in \mathcal{L}(X)$ be Cesàro bounded operator such that $\lim_{n\to\infty} \frac{T^n}{n} x = 0$, $x \in X$. Then T is mean ergodic if and only if $X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}$. Moreover the map $P : X \to X$ defined by $Px := \lim_{n\to\infty} T_n x$ is the projection onto $\mathrm{Ker}(I - T)$ and it satisfies $P = P^2 = TP = PT$.

Mean ergodic properties. Main Theorems

Theorem.

(1) Every power bounded operator on a reflexive Fréchet space is mean ergodic.

(2) Every power bounded operator on a Fréchet Montel space is uniformly mean ergodic.

Theorem. Lin.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(2) $(I - T)(X)$ is closed.

If T a (continuous) operator on a *Fréchet space* X which satisfies $\lim_{n\to\infty} T^n/n = 0$ uniformly on the bounded sets, then (2) implies (1), but in general (1) does not imply (2).

Hypercyclicity. Definitions

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

If the transpose T' of T has two different eigenvalues, then T is not supercyclic.

No power bounded operator can be hypercyclic, but there are mean ergodic hypercyclic operators (Peris).

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

Definition of the Waelbroeck *-spectrum.

X **is a Hausdorff locally convex space (lcs).**

- $\rho^*(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$.
- $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$.
- $\sigma^*(T)$ is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. There exist continuous linear operators T on a Fréchet space X such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly.

Spectrum and mean ergodicity

Theorem. Albanese, Fernández, Galbis, Jordá...

Let X be a Fréchet space. Let $T \in \mathcal{L}(X)$ satisfy

$$\lim_{n\to\infty} \frac{T^n}{n} x = 0, \;\; x \in X.$$

Then $\sigma(T;X) \subset \overline{\mathbb{D}}$.

Theorem. Albanese, Bonet, Ricker.

Let X be a Fréchet space. Let $T \in \mathcal{L}(X)$ be a compact operator such that $1 \in \sigma(T;X)$ with $\sigma(T;X) \setminus \{1\} \subseteq \overline{B(0,\delta)}$ for some $\delta \in (0,1)$ and satisfying $\mathrm{Ker}(I-T) \cap (I-T)(X) = \{0\}$. Then T is both power bounded and uniformly mean ergodic.

An example. Composition operators on $H(\mathbb{D})$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic selfmap on the unit disc. The composition operator is defined by $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_\varphi(f) := f \circ \varphi$ for each $f \in H(\mathbb{D})$.

Theorem. Bonet, Domański.

The following conditions are equivalent:

(1) $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is power bounded.

(2) $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is uniformly mean ergodic.

(3) $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is mean ergodic.

(4) For every compact set $K \subset \mathbb{D}$ there is a compact set $L \in \mathbb{D}$ such that $\varphi^n(K) \subset L$ for each $n \in \mathbb{N}$.

(5) φ has a fixed point in $\mathbb{D}$.

An example. Composition operators on $H(\mathbb{D})$.

Theorem. Bernal, Montes.

The following conditions are equivalent:

(1) $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is hypercylic.

(2) φ is injective and for each compact set $K \subset \mathbb{D}$ there is $n \in \mathbb{N}$ such that $\varphi^n(K) \cap K = \emptyset$.

(3) φ is injective and it does not have fixed points in $\mathbb{D}$.

Second Part

Generalized Cesàro operators on sequence spaces and on weighted Banach spaces of analytic functions on the disc.

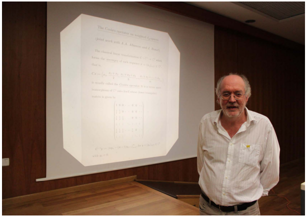
Let us see the Theorems above in action.

We report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The generalized Cesàro operators C_t, for $t \in [0,1]$ acts from $\omega = \mathbb{C}^{\mathbb{N}_0}$ into itself (with $\mathbb{N}_0 := 0, 1, 2, \ldots$) by

$$C_t x := \left(\frac{t^n x_0 + t^{n-1} x_1 + \ldots + x_n}{n+1} \right)_n , \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}_0}. \qquad (2)$$

For $t = 0$ note that C_0 is a diagonal operator with diagonal $\Lambda := (1/(n+1))_n$ and for $t = 1$ that C_1 is the classical Cesàro averaging operator:

$$C_1(x) = \left(\frac{1}{n+1} \sum_{k=0}^{n} x_k \right)_n , \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

The operators C_t for $1 < t < 1$ on sequence spaces were first investigated by Rhaly in the 1980's.

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator C_1 maps the Banach space ℓ^p continuously into itself, and $\|C_1\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

In particular, **Hardy's inequality** holds:

$$\|C_1(x)\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

The operator C_1 is continuous on ℓ_∞ and c_0 with $\|C_1\| = 1$.

Clearly C_1 is not continuous on ℓ_1, since $\mathrm{C}(e_1) = (1, 1/2, 1/3, ...)$.

The operator C_t satisfies $C_t \in \mathcal{L}(\omega)$, and the family of operators $\{C_t : t \in [0,1)\}$ is an equicontinuous subset of $\mathcal{L}(\omega)$.

For each $t \in [0,1]$ the operator $C_t \in \mathcal{L}(\omega)$ is a bicontinuous isomorphism of ω onto itself with inverse operator $(C_t)^{-1} \colon \omega \to \omega$ given by

$$(C_t)^{-1} y = ((n+1)y_n - nty_{n-1})_{n \in \mathbb{N}_0}, \quad y \in \omega \text{ (with } y_{-1} := 0). \tag{3}$$

In particular, C_t is not a compact operator.

Theorem.

For each $t \in [0,1])$ the spectra of $C_t \in \mathcal{L}(\omega)$ are given by

$$\sigma(C_t;\omega) = \sigma_{pt}(C_t;\omega) = \Lambda,$$

with each eigenvalue being simple, and

$$\sigma^*(C_t;\omega) = \Lambda \cup \{0\}.$$

Theorem continued.

The 1-dimensional eigenspace corresponding to the eigenvalue $1/(m+1) \in \Lambda$ is spanned by $x^{[m]}$ with $x_t^{[0]} = (1, t, t^2, ...)$ and

$$x_t^{[m]} := \left(0, \ldots, 0, 1, \frac{(m+1)!}{m!\,1!}t, \frac{(m+2)!}{m!\,2!}t^2, \frac{(m+3)!}{m!\,3!}t^3, \ldots\right), \qquad (4)$$

for each $m \in \mathbb{N}_0$. Observe that $x_t^{[m]} \in \ell_1$.

A similar result holds for $C_1 \in \mathcal{L}(\omega)$.

The discrete generalized Cesàro operator

Proposition. (Sawano, El-Shabrawy)

For each $t \in [0,1)$ the operator $C_t \in \mathcal{L}(\ell^p)$, for $1 \leq p < \infty$, is a compact operator satisfying

$$\|C_t\|_{\ell^1 \to \ell^1} = \frac{1}{t} \log\left(\frac{1}{1-t}\right)$$

and, for $1 < p < \infty$, $t \in (0,1)$,

$$\left(\sum_{n=0}^{\infty} \left(\frac{t^n}{n+1}\right)^p\right)^{1/p} \leq \|C_t\|_{\ell^p \to \ell^p} \leq \left(\frac{1}{t} \log\left(\frac{1}{1-t}\right)\right)^{1/p},$$

with $\|C_0\|_{\ell^p \to \ell^p} = 1$. Moreover,

$$\sigma_{pt}(C_t; \ell^p) = \Lambda \quad \text{and} \quad \sigma(C_t; \ell^p) = \Lambda \cup \{0\}. \tag{5}$$

Proposition.

Let $1 < p < \infty$.

(i) The operator $C_1 \in \mathcal{L}(\ell^p)$ with $\|C_1\|_{\ell^p \to \ell^p} = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) (Leibowitz) The spectra of C_1 are given by

$$\sigma_{pt}(C_1; \ell^p) = \emptyset \quad \text{and} \quad \sigma(C_1; \ell^p) = \left\{ z \in \mathbb{C} : \left| z - \frac{p'}{2} \right| \leq \frac{p'}{2} \right\}.$$

Moreover, the range $(C_1 - zI)(\ell^p)$ is not dense in ℓ^p whenever $|z - \frac{p'}{2}| < \frac{p'}{2}$.

Proposition. (Sawano, El-Shabrawy)

For each $t \in [0,1)$ the operator $C_t \in \mathcal{L}(\ell^\infty)$, $C_t \in \mathcal{L}(c_0)$, and it is a compact operator satisfying $\|C_t\| = 1$.

Moreover,

$$\sigma_{pt}(C_t; \ell^\infty) = \sigma_{pt}(C_t; c_0) = \Lambda,$$

and

$$\sigma(C_t; \ell^\infty) = \sigma(C_t; c_0) = \Lambda \cup \{0\}.$$

The discrete generalized Cesàro operator

Theorem. Leibowitz.

(i) $\sigma(C_1; \ell^\infty) = \sigma(C_1; c_0) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(C_1; \ell^\infty) = \{(1, 1, 1, ...)\}$.

(iii) $\sigma_{pt}(C_1; c_0) = \emptyset$.

The results for C_t were extended by **Curbera and Ricker (in JMAA, 2022)** when the operator C_t acts on other Banach lattices X of sequences.

They investigate the cases when X is the discrete Cesàro space ces_p and its dual d_p, for $1 \leq p \leq \infty$ and $p = 0$, when X is a weighted c_0 or a weighted ℓ_p space $1 \leq p < \infty$, and also Bachelis space $N^p, 1 < p < \infty$.

Together with Albanese and Ricker we investigated (in 2023) the behaviour of the operators C_t **when they act in certain non-normable sequence spaces contained in $\mathbb{C}^{\mathbb{N}_0}$.**

We denote $H(\mathbb{D})$ the space of holomorphic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$ endowed with the topology of uniform convergence on compact subsets, and by H^∞ the Banach space of bounded analytic functions.

For $t \in [0,1)$ we define $C_t \colon H(\mathbb{D}) \to H(\mathbb{D})$, for $f \in H(\mathbb{D})$, by

$$C_t f(0) := f(0)$$

and

$$C_t f(z) := \frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} \, d\xi, \ z \in \mathbb{D} \setminus \{0\}, \tag{6}$$

The operator C_t also has the following representation, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$

$$C_t f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \ldots + a_n}{n+1} \right) z^n,$$

where the coefficients of the series are precisely those of the discrete generalized Cesàro operator.

Moreover, C_0 is given by $C_0 f(z) = \frac{1}{z} \int_0^z f(\xi)\, d\xi$ for $z \neq 0$ and $C_0 f(0) = f(0)$, which is the classical Hardy operator in $H(\mathbb{D})$.

The Cesàro operator for analytic functions

The Cesàro operator C_1 is defined for analytic functions on the disc $\mathbb{D}$ by

$$C_1(f) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$C_1(f)(z) = \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi} \, d\xi, \quad f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

Proposition.

- For every $t \in [0,1]$ the operator $C_t\colon H(\mathbb{D}) \to H(\mathbb{D})$ is continuous. Moreover, the set $\{C_t \,:\, t \in [0,1)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$.
- For each $t \in [0,1]$ the operator $C_t\colon H(\mathbb{D}) \to H(\mathbb{D})$ is an isomorphism and, hence, it is not compact.
- For each $t \in [0,1]$ the spectra of the operator $C_t \in \mathcal{L}(H(\mathbb{D}))$ are given by
$$\sigma_{pt}(C_t; H(\mathbb{D})) = \sigma(C_t; H(\mathbb{D})) = \Lambda \tag{7}$$
and
$$\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0. \tag{8}$$

Proposition.

- C_1 does not act continuously on H^∞.
- For $t \in [0, 1)$ the operator $C_t \colon H^\infty \to H^\infty$ is continuous.
- $\|C_0\|_{H^\infty \to H^\infty} = 1$.
- $\|C_t\|_{H^\infty \to H^\infty} = -\frac{\log(1-t)}{t}, \quad t \in (0, 1)$.
- Moreover, for $t \in [0, 1)$, $C_t \colon H^\infty \to H^\infty$ is compact.

The generalized Cesàro operator for analytic functions

Proposition.

Let $t \in [0,1)$. Then

- For each $t \in [0,1)$ the operator $C_t \colon A(\mathbb{D}) \to A(\mathbb{D})$ is continuous.
- The operator norms satisfy

$$\|C_0\|_{A(\mathbb{D})\to A(\mathbb{D})} = \|C_t\|_{H^\infty \to H^\infty}$$

- $C_t \colon A(\mathbb{D}) \to A(\mathbb{D})$ is compact and $C_t(H^\infty) \subseteq A(\mathbb{D})$.
- $\sigma_{pt}(C_t; H^\infty) = \sigma_{pt}(C_t; A(\mathbb{D}))$.
- $\sigma(C_t; H^\infty) = \sigma(C_t; A(\mathbb{D}))$.

More general results about Volterra operators are due to Contreras, Peláez, Pommerenke and Rättyä (2016).

The weighted Banach spaces

A **weight** v is a continuous function $v : [0,1[\to]0,\infty[$, which is non-increasing on $[0,1[$ and satisfies $\lim_{r\to 1} v(r) = 0$. We extend v to $\mathbb{D}$ by $v(z) := v(|z|)$.

The **weighted Banach spaces of holomorphic functions on $\mathbb{D}$ are defined by**

$$H_v^\infty := \{f \in H(\mathbb{D}) \mid \|f\|_v := \sup_{z\in\mathbb{D}} v(|z|)|f(z)| < \infty\},$$
$$H_v^0 := \{f \in H(\mathbb{D}) \mid \lim_{|z|\to 1} v(|z|)|f(z)| = 0\},$$

and they are endowed with the weighted sup norm $\|\cdot\|_v$.

- H_v^0 is a closed subspace of H_v^∞. And H_v^∞ is canonically the bidual of H_v^0.
- The polynomials are contained and dense in H_v^0 but the monomials do not in general form a Schauder basis (**Lusky**).
- The Cesàro means of the Taylor polynomials satisfy $\|C_n f\|_v \leq \|f\|_v$ for each $f \in H_v^\infty$ and the sequence $(C_n f)_n$ is $\|\cdot\|_v$-convergent to f when $f \in H_v^0$.
- Spaces of type H_v^∞ and H_v^0 appear in the study of growth conditions of analytic functions and have been investigated by many authors since the work of Shields and Williams in 1971. Classical operators on these spaces have been also investigated thoroughly.

The generalized Cesàro operator on weighted Banach spaces

Theorem.

(1) Let v be a weight function on $[0,1)$. For each $t \in [0,1)$ the operator $C_t\colon H_v^\infty \to H_v^\infty$ is continuous. Moreover, $\|C_0\|_{H_v^\infty \to H_v^\infty} = 1$ and

$$1 \leq \|C_t\|_{H_v^\infty \to H_v^\infty} \leq -\frac{\log(1-t)}{t}, \quad t \in (0,1).$$

(2) For each $t \in [0,1)$, the operator $C_t\colon H_v^0 \to H_v^0$ is continuous and satisfies $\|C_t\|_{H_v^0 \to H_v^0} = \|C_t\|_{H_v^\infty \to H_v^\infty}$.

Moreover, the operators $C_t\colon H_v^\infty \to H_v^\infty$ and $C_t \to H_v^0 \to H_v^0$ are compact.

The generalized Cesàro operator on weighted Banach spaces

Theorem.

Let v be a weight function on $[0,1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$. For each $t \in [0,1)$ the spectra of $C_t \in \mathcal{L}(H_v^\infty)$ and of $C_t \in \mathcal{L}(H_v^0)$ are given by

$$\sigma_{pt}(C_t; H_v^\infty) = \sigma_{pt}(C_t; H_v^0) = \left\{ \frac{1}{m+1} \ : n \in \mathbb{N}_0 \right\}, \tag{9}$$

and

$$\sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \left\{ \frac{1}{m+1} \ : n \in \mathbb{N}_0 \right\} \cup \{0\}. \tag{10}$$

The generalized Cesàro operator on weighted Banach spaces

Let $n \in \mathbb{N}$ be fixed. Consider the weight $v(z) = (\log \frac{e}{1-|z|})^{-n}$, for $z \in \mathbb{D}$, which satisfies $v(0) = 1$ and $\lim_{|z|\to 1^-} v(z) = 0$.

The function $f(z) := [\log(1-z)]^n \in H(\mathbb{D})$ belongs to H_v^∞. But,

$$C_1 f(z) = \frac{1}{z}\int_0^z \frac{(\log(1-\xi)^n)}{1-\xi} d\xi = -\frac{1}{(n+1)z}(\log(1-z))^{n+1}, \quad z \in \mathbb{D}.$$

Accordingly, $C_1 f \notin H_v^\infty$.

The Cesàro operator C_1 is not well-defined on H_v^∞, that is, $C_1(H_v^\infty) \not\subseteq H_v^\infty$. However, $C_t \in \mathcal{L}(H_v^\infty)$ for every $t \in [0,1)$.

We consider the case of the standard weights $v_\gamma(z) := (1-|z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$.

Aleman and Persson (2008-10) showed that C_1 is continuous on $H^\infty_{v_\gamma}$ and they calculated the spectrum and point spectrum in this case.

Proposition.

Let $t \in (0,1)$ and $\gamma > 0$.

(i) The operator norm $\|C_t\|_{H^\infty_{v_\gamma} \to H^\infty_{v_\gamma}} = 1$, for every $\gamma \geq 1$.

(ii) For each $\gamma \in (0,1)$, the inequality $\|C_t\|_{H^\infty_{v_\gamma} \to H^\infty_{v_\gamma}} \leq \min\{-\frac{\log(1-t)}{t}, \frac{1}{\gamma}\}$ is valid.

Arroussi, Gissy and Virtanen (2023) have proved that C_1 is continuous on H^∞_v for exponential weights $v(r) = \exp(-\alpha/(1-r)^\beta)$.

The norm of C_1. Standard weights

Aleman and Persson gave no quantitative estimate for the operator norm of $C_1 : H^\infty_{v_\gamma} \to H^\infty_{v_\gamma}$.

Theorem.

(i) Let $\gamma \geq 1$. Then $\|(C_1)^n\| = 1$ for all $n \in \mathbb{N}$.

(ii) Let $0 < \gamma < 1$. Then $\|(C_1)^n\| = 1/\gamma^n$ for all $n \in \mathbb{N}$.

The same equalities hold for $C_1 : H^0_{v_\gamma} \to H^0_{v_\gamma}$.

The spectrum of the classical Cesàro operator. Standard weights.

Theorem. Aleman, Persson.

Let $\gamma > 0$. The Cesàro operator $C_1 \colon H^0_{v_\gamma} \to H^0_{v_\gamma}$ has the following properties.

(i) $\sigma_{pt}(C_1, H^0_{v_\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(C_1, H^0_{v_\gamma}) = \sigma_{pt}(C_1, H^0_{v_\gamma}) \cup \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) If $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ (equivalently $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$), then $\mathrm{Im}(\lambda I - C_1)$ is a closed subspace of $H^0_{v_\gamma}$ and has codimension 1.

The spectrum of the classical Cesàro operator. Standard weights.

Theorem. Aleman, Persson.

Moreover, the Cesàro operator $C_1 : H^\infty_{v_\gamma} \to H^\infty_{v_\gamma}$ satisfies

(iv) $\sigma_{pt}(C_1, H^\infty_{v_\gamma}) = \{\frac{1}{m} : m \in \mathbb{N},\ m \leq \gamma\}$, and

(v) $\sigma(C_1, H^\infty_{v_\gamma}) = \sigma(C_1, H^0_{v_\gamma})$.

The generalized Cesàro operator on weighted Banach spaces

Recall that the Cesàro operator C_1 is not well-defined on H_v^∞ for the weight $v(z) = (\log(\frac{e}{1-|z|}))^{-n}, \ z \in \mathbb{D}$.

Theorem.

(1) Let v be a weight function on $[0,1)$ such that $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty \to H_v^\infty} < \infty$. Then $C_1 \in \mathcal{L}(H_v^\infty)$.

(2) For each $n \in \mathbb{N}$, let $v(z) = (\log(\frac{e}{1-|z|}))^{-n}$ for $z \in \mathbb{D}$. Then $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty \to H_v^\infty} = \infty$.

Proposition.

Let $t \in [0,1)$ and $x^{[0]} := (t^n)_{n\in\mathbb{N}_0}$. The generalized Cesàro operator $C_t \in \mathcal{L}(\omega)$ is power bounded and uniformly mean ergodic. Moreover, $\mathrm{Ker}(I - C_t) = \mathrm{span}\,\{x^{[0]}\}$ and the range

$$(I - C_t)(\omega) = \{x \in \omega : \ x_0 = 0\} = \overline{\mathrm{span}\,\{e_r : \ r \in \mathbb{N}\}} \tag{11}$$

of $(I - C_t)$ is closed in ω. The operator C_t is not supercyclic in ω.

$C_1 \in \mathcal{L}(\omega)$ is also power bounded, uniformly mean ergodic and not supercyclic.

Proposition.

- Let $t \in [0,1)$. Let X belong to any one of the sets: $\{\ell^p : 1 \leq p < \infty\}$ or c_0. Then $C_t \in \mathcal{L}(X)$ is power bounded and uniformly mean ergodic, but not supercyclic.
- The Cesàro operator $C_1 \in \mathcal{L}(\ell^p), 1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.
- The Cesàro operator $C_1 \in \mathcal{L}(c_0)$ is power bounded, not mean ergodic and not supercyclic.

Ergodicity of C_t on spaces of holomorphic functions.

Proposition.

For each $t \in [0, 1)$ the operator $C_t \colon H(\mathbb{D}) \to H(\mathbb{D})$ is power bounded, uniformly mean ergodic but, it fails to be supercyclic. Moreover, $(I - C_t)(H(\mathbb{D}))$ is the closed subspace of $H(\mathbb{D})$ given by

$$(I - C_t)(H(\mathbb{D})) = \{g \in H(\mathbb{D}) \,:\, g(0) = 0\} \tag{12}$$

and we have the decomposition

$$H(\mathbb{D}) = \mathrm{Ker}(I - C_t) \oplus (I - C_t)(H(\mathbb{D})). \tag{13}$$

The Cesàro operator C_1 acting on $H(\mathbb{D})$ is power bounded, uniformly mean ergodic and not supercyclic.

Ergodicity of C_t on spaces of holomorphic functions.

Proposition.

For each $t \in [0,1)$ both of the operators $C_t \in \mathcal{L}(H^\infty)$ and $C_t \in \mathcal{L}(A(\mathbb{D}))$ are power bounded, uniformly mean ergodic but, fail to be supercyclic.

Proposition.

Let v be a weight function on $[0,1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$.

For each $t \in [0,1)$ both of the operators $C_t \in \mathcal{L}(H_v^\infty)$ and $C_t \in \mathcal{L}(H_v^0)$ are power bounded, uniformly mean ergodic and fail to be supercyclic.

Moreover, $\mathrm{Ker}(I - C_t) = \mathrm{span}\{g_0\}$, with $g_0(z) = \sum_{n=0}^{\infty} t^n z^n$, for $z \in \mathbb{D}$, and $\mathrm{Im}(I - C_t) \subseteq \{g \in H_v^\infty \,:\, g(0) = 0\}$

Ergodicity of C_1. Standard weights.

The standard weights are $v_\gamma(z) := (1-|z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$. As mentioned before $C_1 \in \mathcal{L}(H^\infty_{v_\gamma})$ and $C_1 \in \mathcal{L}(H^0_{v_\gamma})$.

Theorem.

(i) Let $0 < \gamma < 1$. The operators $C_1 \in \mathcal{L}(H^\infty_{v_\gamma})$ and $C_1 \in \mathcal{L}(H^0_{v_\gamma})$ are not power bounded and not mean ergodic. Moreover, $\mathrm{Ker}(I - C_1) = \{0\}$, and $\mathrm{Im}(I - C_1)$ is a proper closed subspace of $H^\infty_{v_\gamma}$.

(ii) For $\gamma = 1$, the operators $C_1 \in \mathcal{L}(H^\infty_{v_1})$ and $C_1 \in \mathcal{L}(H^0_{v_1})$ are power bounded but not mean ergodic. Moreover, $\mathrm{Im}(I - C_1)$ is not a closed subspace of $H^\infty_{v_\gamma}$.

Ergodicity of C_1. Standard weights.

Theorem continued

(iii) Let $\gamma > 1$. Both of the operators $C_1 \in \mathcal{L}(H^\infty_{v_\gamma})$ and $C_1 \in \mathcal{L}(H^0_{v_\gamma})$ are power bounded and uniformly mean ergodic. In addition, $\mathrm{Im}(I - C_1)$ is a proper closed subspace of $H^\infty_{v_\gamma}$. And

$$\mathrm{Im}(I - C_1) = \{h \in H^\infty_{v_\gamma} : h(0) = 0\}.$$

Moreover, with $\varphi(z) := 1/(1-z)$, for $z \in \mathbb{D}$, the linear projection operator $P_\gamma : H^\infty_{v_\gamma} \to H^\infty_{v_\gamma}$ given by

$$P_\gamma(f) := f(0)\varphi, \qquad f \in H^\infty_{v_\gamma},$$

is continuous and satisfies $\lim_{n\to\infty}(C_1)_{[n]} = P_\gamma$ in the operator norm.

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