On spaces ℓ_∞ and c_0 with the pointwise topology embedded into spaces $C_p(X)$

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Let us recall a few results regarding Banach spaces $C(X)$ of continuous real valued functions on a compact Hausdorff spaces X related to containing copies of the Banach spaces c_0 or ℓ_∞ which will motivate further results on this topic but for the spaces $C_p(X)$.

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We will see...

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5. (CH) There is compact X without $\beta\mathbb{N}$ such that $C(X)$ does not have a quotient isomorphic to ℓ_∞ (Talagrand).

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6. $((MA) \wedge \sim (CH))$ Every nonreflexive Grothendieck space has quotient ℓ_∞ (Haydon-Levy-Odell).

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A Banach space $C(X)$ is Grothendieck iff $C(X)$ does not contain a complemented copy of c_0.

Theorem 2 (Cembranos-Freniche)

For every infinite compact spaces K and L the Banach space $C(K \times L)$ contains a complemented copy of the space c_0.

1. To study C_p variants of the above results, define $(c_0)_p = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : x_n \to 0\}$, $(\ell_\infty)_p = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$.

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2. $C_p(X)$ satisfies the ***Josefson-Nissenzweig property*** (JNP) if there is a sequence (μ_n) of finitely supported sign-measures on X with $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$, and $\mu_n(f) \to_n 0$ for each $f \in C_p(X)$ (Banakh-Kąkol-Śliwa),

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3. where $C_p(X)' \ni \mu = \sum_{x \in F} a_x \delta_x$ for some finite $F \subset X$, $\|\mu\| = \sum_{x \in F} |a_x|$.
4. **Easy fact**: If a compact space X contains a non-trivial convergent sequence, then $C_p(X)$ satisfies the JNP.

Theorem 3 (Banakh-Kąkol-Śliwa)

For a Tychonoff space X the following are equivalent:

1. $C_p(X)$ *contains a complemented copy of* $(c_0)_p$.
2. $C_p(X)$ *has a quotient isomorphic to* $(c_0)_p$.
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This provides a C_p-variant of the well known theorem stating that **for a compact space X the Banach space $C(X)$ contains a complemented copy of c_0 if and only if $C(X)$ admits a continuous linear surjection onto c_0.**

Theorem 4 (Kąkol-Molto-Śliwa)

Let X be an infinite Tychonoff space. Then $C_p(X)$ contains a copy of $(c_0)_p$. If X contains an infinite compact subset, then $C_p(X)$ contains a closed copy of $(c_0)_p$.

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Corollary 5

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Theorem 8 (Kąkol–Marciszewski-Sobota-Zdomskyy)

It is consistent that there exists an infinite pseudocompact space X such that $C_p(X \times X)$ does not contain a complemented copy of $(c_0)_p$.

1. $C(K)$ has *the* ***Grothendieck property*** *for a family* $\mathcal{M} \subset C(K)^*$ *of measures* if every weak* null sequence $(\mu_n) \subset \mathcal{M}$ is weakly null (Kąkol-Sobota-Zdomskyy).

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2. $C(K)$ is $\ell_1(K)$-***Grothendieck***, if it has a Grothendieck property for $\mathcal{M} = \ell_1(K) \subset C(K)^*$, where $\gamma \in \ell_1(K)$ acts on $C(K)$ as $\langle f, \gamma \rangle = \sum_{x \in K} f(x) \gamma(x)$ since $\sum_{x \in K} |\gamma(x)| < \infty$. Recall $\ell_1(K)$ is isomorphic $M_d(K)$ - discrete measures on K.

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Theorem 9 (Kąkol-Sobota-Zdomskyy)

Let K be an infinite compact space.

1. *If $C(K)$ is a Grothendieck space, $C_p(K)$ fails the JNP.*
2. *$C(K)$ is $\ell_1(K)$-Grothendieck iff $C_p(K)$ fails the JNP.*

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Theorem 10

1. *The space $(\ell_\infty)_p$ is σ-compact while $(c_0)_p$ is not.*
2. *There is a continuous map from $(c_0)_p$ onto $(\ell_\infty)_p$ but no linear continuous surjection between both spaces exists.*
3. *Let X be a compact space* ***containing*** *$\beta\mathbb{N}$. Then $C_p(X)$ admits a continuous open linear map onto $(\ell_\infty)_p$ (Banakh-Kąkol-Śliwa).*
4. *(CH) there exists a compact space X* ***not containing*** *$\beta\mathbb{N}$ such that $C(X)$ is Grothendieck and there is no continuous linear surjection from $C_p(X)$ onto $(\ell_\infty)_p$.*

1. Recall two motivating facts: For a Tychonoff space X the space $C_p(X)$ **is** σ**-compact iff** X **is finite** (Velichko) **iff** $C_p(X)$ **has a fundamental sequence of bounded sets** (Ferrando-Kąkol).

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Let X be a Tychonoff space containing a non scattered compact subspace. Then $C_p(X)$ contains a closed σ-compact infinite-dimensional subspace.

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3. Summing up we have even the following

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For an infinite compact space X the following are equivalent:

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Corollary 16

For infinite compact scattered space X the closure of the σ-compact subspace $(\ell_\infty)_p$ of $C_p(X)$ is not σ-compact.

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Example 17

Let F be a set with $|F| \geq \mathfrak{d}$ endowed with the discrete topology and let X be the product $F \times [1, \omega]$. Then there is a sequence $f_1, f_2, \ldots$ in $C_p(X)$ that forms an algebraic basis of a closed σ-compact subspace of $C_p(X)$.