Dynamical properties admitted by the Lelek fan

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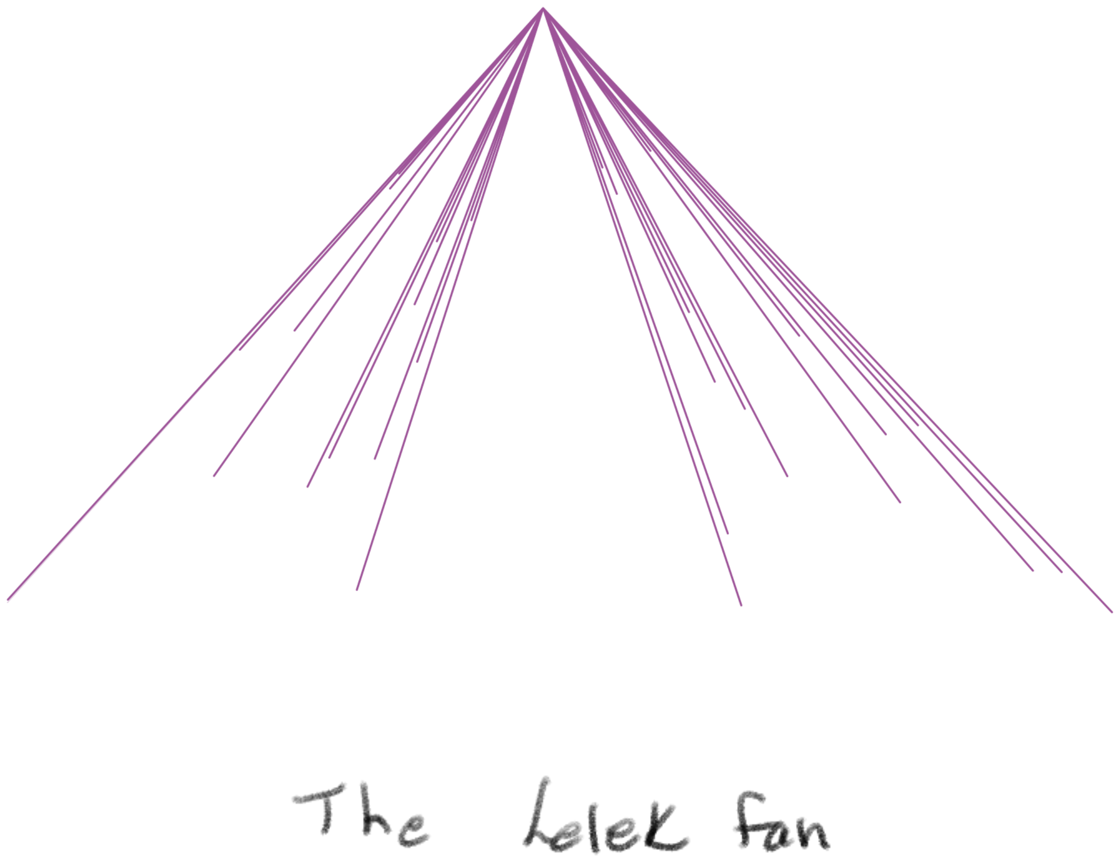
Recently, many researchers in topological dynamics have been investigating the dynamical properties admitted by non-manifold continua. Such continua include the fans, dendrites, dendroids, and indecomposable continua. This work gives insight into both the topology of a continuum and the dynamical properties it admits as well as how the topology and dynamics interact. In particular, we would like to know: How simple can a continuum be and still admit complicated dynamics?

Here we focus on the Lelek fan.

Lelek fan

The Lelek fan was constructed by A. Lelek in 1960. After that, several characterizations of the Lelek fan were presented by J. J. Charatonik, W. J. Charatonik and S. Miklos in 1989/1990. An interesting property of the Lelek fan X is the fact that the set of its end-points is a dense one-dimensional set in X. It is also unique, i.e., it is the only non-degenerate smooth fan with a dense set of end-points. This was proved independently by W. D. Bula and L. Oversteegen and by W. Charatonik in 1989/1990. They proved that any two non-degenerate subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic.


The Lelek fan

(1) We use Mahavier products of closed relations on the square $[0,1]\times[0,1]$ that generate Lelek fans. (The relation is a pair of straight line segments both of which contain $(0,0)$, one of which has a point on $[0,1]\times\{1\}$, the other of which has a point on $\{1\}\times[0,1]$ and chosen so that the two lines "never connect".)

(2) With this construction we show that the shift map on the Mahavier product (a) has positive topological entropy, (b) has sensitive dependence on initial conditions, (c) admits transitive homeomorphisms as well as transitive non-invertible maps, (d) admits mixing homeomorphisms as well as mixing non-invertible maps, and (e) admits homeomorphisms and non-invertible maps that are chaotic in the sense of Robinson, but not Devaney.

The papers involved are the following. All are available on the ArXive.

(1) (Banič, Erceg, K), Closed relations with non-zero entropy that generate no periodic points, DCDS **22** 2022, 5137 - 5166.
(2) (Banič, Erceg, K), Lelek fan as an inverse limit with a single set-valued bonding function, to appear, Mediterranean Journal of Mathematics.
(3) (Banič, Erceg, K), A transitive homeomorphism on the Lelek fan, Journal of Difference Equations Applications **29** 2023
(4) (Banič, Erceg, K, Mouron, Nall), Chaos and mixing homeomorphisms on fans, submitted
(5) (Banič, Erceg, Gril Rogina, K, Mouron, Nall) Sufficient conditions for non-zero entropy of closed relations, to appear, ETDS

Definitions and Notation

Definition

Let X be a compact metric space and let $G \subseteq X \times X$ be a relation on X. If $G \in 2^{X \times X}$, then we say that G is *a closed relation on X*.

Definition

Let X be a compact metric space and let G be a closed relation on X. Then we call

$$I_G^+ = \star_{i=1}^{\infty} G = \left\{ (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \mid \text{ for each } i, (x_i, x_{i+1}) \in G \right\}$$

the Mahavier product of G. (New notation: we also use I_G^+ to denote $\star_{i=1}^{\infty} G$

Definition

Let X be a compact metric space and let G be a closed relation on X. The function

$$\sigma : I_G^+ \to I_G^+,$$

defined by

$$\sigma(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, x_4, \ldots) \in \star_{n=1}^{\infty} G$, is called *the shift map on* $\star_{n=1}^{\infty} G = I_G^+$.

Definition Let (X, f) be a dynamical system. We say that (X, f) has *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $d(x, y) > \delta$.

Definition Let X be a compact metric space, F be a closed relation on X, and α be an open cover of X. The *entropy of F with respect to the open cover α* is

$$ent(F, \alpha) = \lim_{m \to \infty} \frac{\log N(X_F^m)}{m}.$$

The *entropy* of F is defined by

$$ent(F) = \sup_{\alpha} ent(F, \alpha).$$

Let (X, f) be a dynamical system. We say that (X, f) is
(1) *transitive*, if for all non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$, and
(2) *dense orbit transitive*, if there is a point $x \in X$ such that its orbit $\{x, f(x), f^2(x), f^3(x), \dots\}$ is dense in X.

Let (X, f) be a dynamical system. We say that the mapping f is *transitive*, if (X, f) is transitive.

Observation It is a well-known fact that if X has no isolated points, then (X, f) is transitive if and only if (X, f) is dense orbit transitive.

Theorem

Let (X, f) be a dynamical system. If (X, f) is transitive and $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$ is the shift map on $\varprojlim(X, f)$, then also $(\varprojlim(X, f), \sigma)$ is a transitive dynamical system.

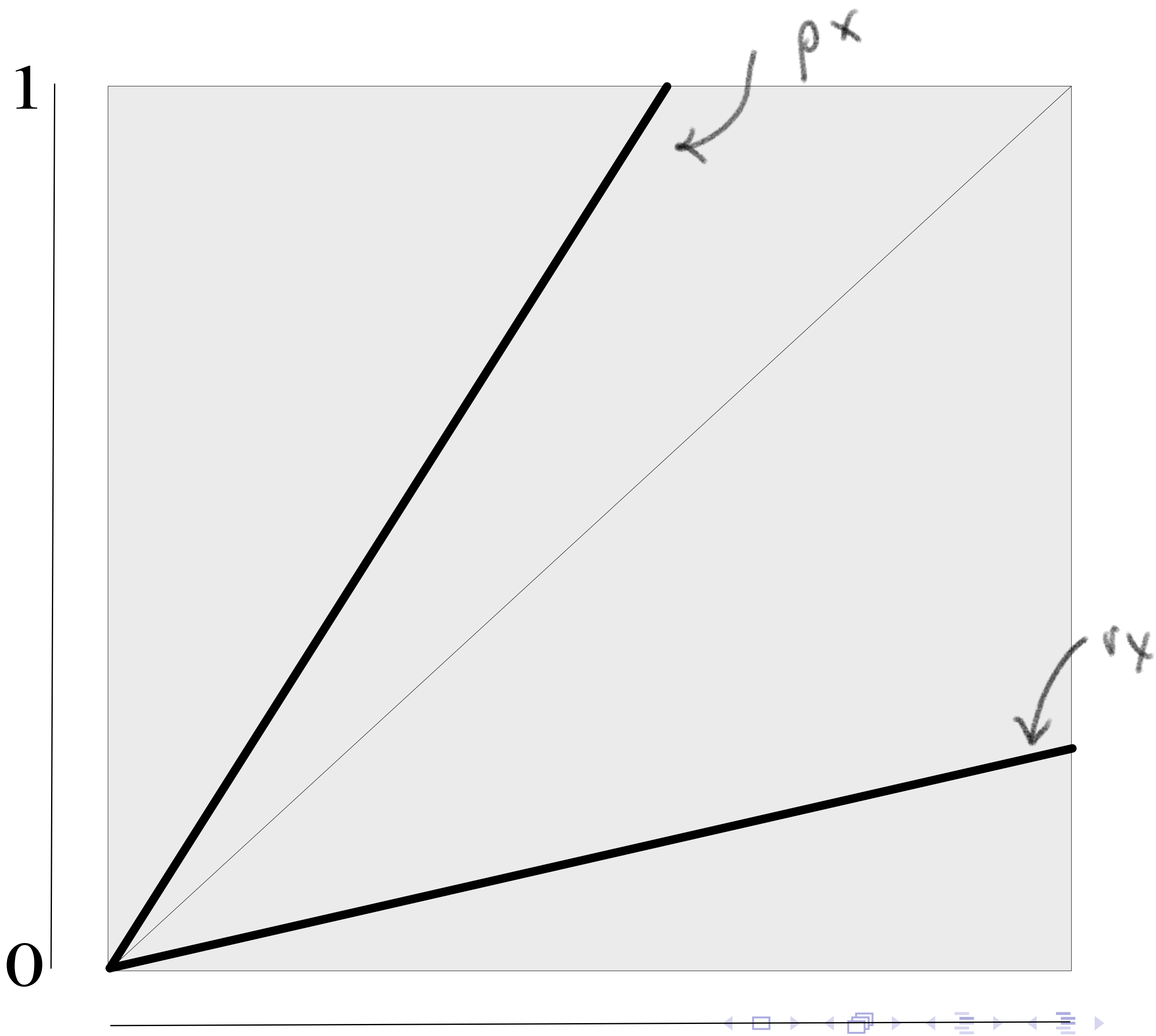
Definition Let (X, f) be a dynamical system. We say that (X, f) is *mixing* if for all nonempty open sets U and V in X, there is a non-negative integer n_0 such that for each positive integer $n \geq n_0$, $f^n(U) \cap V$ is not empty.

Robinson's Chaos and Devaney's Chaos

Definition Let (X, f) be a dynamical system. We say that (X, f) is *chaotic in the sense of Robinson* if (X, f) is transitive, and has sensitive dependence on initial conditions.

Definition Let (X, f) be a dynamical system. We say that (X, f) is *chaotic in the sense of Devaney* if (X, f) is transitive, has sensitive dependence on initial conditions, and has a dense set of periodic points.

Lelek fan as a Mahavier product of two line segments



Definition

For each positive integer k, we use $\pi_k : \prod_{i=1}^{\infty}[0,1] \to [0,1]$ to denote *the k-th standard projection* from $\prod_{i=1}^{\infty}[0,1]$ to $[0,1]$: for each $(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty}[0,1]$,

$$\pi_k(x_1, x_2, x_3, \ldots) = x_k.$$

In our first paper we gave sufficient conditions for a closed relation on I to have non-zero entropy.

Definition Suppose $0 < r < 1$ and $\rho > 1$. Then r, ρ *never connect* ($(r, \rho) \in NC$) if when $k, l, k', l' \geq 0$, $(k, l) \neq (k', l')$, then $r^k \rho^l \neq r^{k'} \rho^{l'}$.

For $M_{r\rho}$ with $(r, \rho) \in NC$, $M_{r\rho}$ has positive entropy but no periodic points other than the top. (So no Bernoulli shifts happen.)

First we showed that $\{r^k \rho^l \leq 1 : k, l \geq 0\}$ is dense in $[0, 1]$. (This was the start of real understanding of the two lines example.) The two lines relation generates a fan $M_{r\rho}$ (not hard to believe but we proved it.)

Definition If X is a fan, let
$E(X) = \{e \in X : e$ is the non-top endpoint of a leg $\in X\}$.

Theorem Let $(r, \rho) \in NC$. If $0 < x < 1$, then there is a sequence $a_1, a_2, \cdots$ of $r's$ and $\rho's$ such that for each n. $a_1 a_2 \cdots a_n x < 1$ and *lim sup* $a_1 a_2 \cdots a_n = 1$. Furthermore,
$(x, a_1 x \cancel{<}, a_2 a_1 x, \cdots \in E(M_{r\rho}).$

Theorem $E(M_{r\rho})$ is dense in $M_{r\rho}$. From this, it follows that $M_{r\rho}$ is a Lelek fan.

Lemma There is a countable subcollection $\mathcal{U}$ of open sets in I^∞ such that

long lemma

1. Each $U \in \mathcal{U}$ intersects $M_{r\rho}$.
2. If $U \in \mathcal{U}$, then there exists n such that $U = U_1 \times U_2 \times \cdots U_n \times I^\infty$ such that each U_i is an open interval in $(0,1)$.
3. If $U \in \mathcal{U}$, then there exists a finite sequence $a_1, \cdots, a_n$ of r's and ρ's such that $U_{i+1} = a_i U_i$ for $1 < i < n$ and if $\mathbf{x} \in M_{r\rho}$, and $b_1, b_2, \cdots$ is the sequence of r's and ρ's associated with $\mathbf{x}$, then $a_i = b_i$ for $i = 1, \cdots, n-1$.
4. For every $\mathbf{z} \in M_{r\rho}$ with $z_i \neq 0, z_i \neq 1$, for any i and for every $\epsilon > 0$ there exists $U \in \mathcal{U}$ such that $\mathbf{z} \subset U \subset B_\epsilon(\mathbf{z})$.

σ-transitive

If there exists for I and a relation R on I that admits such a collection $\mathcal{U}$, we say that σ_R^+ on I_R^+ is σ-transitive. This has been generalized to more general compact metric spaces.

We have also studied σ-transitivity of dynamical systems with closed relations. One of the results we prove is that σ-transitivity of (X,F) is equivalent to the transitivity of the dynamical system (X_F^+,σ_F^+).

Theorem $M_{r\rho}$ with the shift $\sigma_L \cancel{r,\rho}^+$ admits non-invertible transitive maps.

r,ρ

But $M_{r\rho}$ also admits transitive homeomorphisms.

Definition 5.1. We use M to denote the inverse limit

$$M = \varprojlim (M_{r,\rho}, \sigma_{r,\rho})$$

and we use σ to denote the shift map

$$\sigma : M \to M$$

on M.

Observation 5.2. *M is a continuum since it is an inverse limit of continua.*

Let $I_L = \{(\ldots x_{-2}, x_{-1}; x_0, x_1, \ldots) : \forall i, (x_i, x_{i+1}) \in L\}$.

Then $I_L \cong M$.

Thm. If $\sigma_{r,\rho} : M_{r,\rho} \to M_{r,\rho}$ is transitive, then $\sigma : M \to M$ is also transitive.

Thm. M is a Lelek fan.

$\therefore$ $\sigma : M \to M$ is a transitive homeomorphism on a Lelek fan.

Thm. If F is a closed relation on I, then $\sigma^+ : I_F^+ \to I_F^+$ is transitive iff $\sigma : I_F \to I_F$ is transitive.

What about other dynamical properties on $I^+_{L_{r,p}}$ and $I_{L_{r,f}}$?

We have looked at sensitive dependence on initial conditions and mixing:

Observation 2.23. *Let (X, f) be a dynamical system. Note that (X, f) has sensitive dependence on initial conditions if and only if there is $\varepsilon > 0$ such that for each non-empty open set U in X, there is a positive integer n such that* $\mathrm{diam}(f^n(U)) > \varepsilon$. *See [5, Theorem 2.22] for more information.*

Definition 2.24. *Let (X, f) be a dynamical system and let A be a non-empty closed subset of X. We say that (X, f)* has sensitive dependence on initial conditions with respect to A, *if there is $\varepsilon > 0$ such that for each non-empty open set U in X, there are $x, y \in U$ and a positive integer n such that*

$$\min\{d(f^n(x), f^n(y)), d(f^n(x), A) + d(f^n(y), A)\} > \varepsilon.$$

Proposition 2.25. *Let (X, f) be a dynamical system and let A be a non-empty closed subset of X. If (X, f) has sensitive dependence on initial conditions with respect to A, then (X, f) has sensitive dependence on initial conditions.*

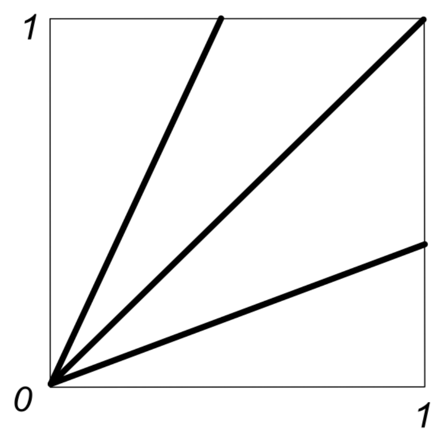
Thm (I_L^+, σ^+) and (I_L, σ_L) have sensitive dependence on initial conditions.

Consider:

$$F_{r,\rho} = L_{r,\rho} \cup \{(t,t) \mid t \in X\}$$

← new relation

$L_{r,\rho}$ plus diagonal



<u>Def.</u> Let (X, f) be a dynamical system. Then (X, f) is <u>mixing</u> if for all nonempty open sets U and V in X, there is a non-negative $n_0 \ni \forall$ p.i.i. n, $n \geq n_0 \Rightarrow f^n(U) \cap V \neq \emptyset$.

<u>Thm.</u> $(I^+_{F_{r,p}}, \sigma^+_{F_{r,p}})$ and $(I_{F_{r,p}}, \sigma_{r,p})$ are mixing.

$I^+_{F_{r,p}}$ and $I_{F_{r,p}}$ are Lelek fans.

Thm. Let G be a closed relation on $\mathbb{X}$, and let $F = G \cup \Delta_{\mathbb{X}}$ and let $p_1(G) = p_2(G) = \mathbb{X}$.

Then

(a) if $(\mathbb{X}_G^+, \sigma_G^+)$ is transitive, so is $(\mathbb{X}_F^+, \sigma_F^+)$;

(b) if $(\mathbb{X}_G, \sigma_G)$ is transitive, so is $(\mathbb{X}_G, \sigma_G)$.

(c) if $(\mathbb{X}_G^+, \sigma_G^+)$ is transitive, $(\mathbb{X}_F^+, \sigma_F^+)$ is mixing.

Two lines example: $(I^+_{L_{r,\rho}}, \sigma^+_{r,\rho})$ has chaos in the sense of Robinson, but not in sense of Devaney

Thanks so much for listening!