Return time sets and product recurrence

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Topological dynamical system

Definition

By a **topological dynamical system**, we mean a pair (X, T), where X is a compact metric space with a metric d, and T is a continuous map from X to itself.

Convention: $T^0 = id_X$, $T^n = T \circ T^{n-1}$, $\forall n \in \mathbb{N}$.

For a point $x \in X$ and a subset U of X, the **return time set of x into U** is

$$N(x, U) = \{n \in \mathbb{N}_0 : T^n x \in U\}.$$

Return time set

The study of return time set plays an important role in the following aspects:

1. Classifications of topological dynamical systems.
2. Characterizations of dynamics properties.
3. Applications to combinatorial number theory.

IP-set

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Definition

Let $\{p_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}_0$. The **finite sum** of $\{p_i\}_{i=1}^{\infty}$ is

$$FS(\{p_i\}_{i=1}^{\infty}) = \left\{ \sum_{i\in\alpha} p_i : \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\}.$$

We say that a subset F of $\mathbb{N}_0$ is an **IP-set** if there exists a sequence $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{N}_0$ such that $FS(\{p_i\}_{i=1}^{\infty})$ is infinite and contained in F.

IP-set and recurrence

We say that a point $x \in X$ is **recurrent** if for every neighborhood U of x, the return time set $N(x, U)$ is infinite.

Theorem (Furstenberg 1981)

1. *Let (X, T) be a topological dynamical system and $x \in X$. If x is recurrent, then for every neighborhood U of x, the return time set $N(x, U)$ is an IP-set.*
2. *For every IP-subset F of $\mathbb{N}_0$, there exists a topological dynamical system (X, T), a recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F \cup \{0\}$.*

Furstenberg, H. Recurrence in ergodic theory and combinatorial number theory. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1981.

Almost periodic point

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Definition

We say that a subset F of $\mathbb{N}_0$ is **syndetic** if there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}_0$, $F \cap [n, n+N] \neq \emptyset$.

Let (X, T) be a topological dynamical system. We say that a point $x \in X$ is **almost periodic** if for every neighborhood U of x, the return time set $N(x, U)$ is syndetic.

Theorem (Birkhoff 1927, Gottschalk-Hedlund 1955)

Let (X, T) be a topological dynamical system. A point $x \in X$ is almost periodic if and only if $(\overline{\mathrm{Orb}(x, T)}, T)$ is minimal, that is $\overline{\mathrm{Orb}(x, T)}$ does not contain any proper T-invariant closed subsets.

Distal point

Definition

Let (X,T) be a topological dynamical system and $x, y \in X$. If $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$, then we say that (x, y) is **proximal**.

A point $x \in X$ is called **distal** if for any $y \in \overline{\mathrm{Orb}(x,T)} \setminus \{x\}$, (x, y) is not proximal, that is, $\liminf_{n\to\infty} d(T^n x, T^n y) > 0$.

We say that (X,T) is **distal** if any point in X is distal.

Distal point

Definition

We say that a subset F of $\mathbb{N}_0$ is an **IP*-set** if for every IP-set H in $\mathbb{N}_0$, $F \cap H \neq \emptyset$.

Theorem (Furstenberg 1981)

Let (X,T) be a topological dynamical system and $x \in X$. Then the following assertions are equivalent:

1. *x is a distal point;*
2. *for every neighborhood U of x, the return time set $N(x,U)$ is an IP*-set;*
3. *x is* **product recurrent***, that is, for every dynamical system (Y,S) and every recurrent point $y \in Y$, (x,y) is recurrent in the product system $(X \times Y, T \times S)$.*

Definition

We say that a subset F of $\mathbb{N}_0$ is a **central set** if there exists a topological dynamical system (X,T), a point $x \in X$, an almost periodic point $y \in X$ and a neighborhood U of y such that

1. (x,y) is proximal, that is, $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$
2. $N(x,U) \subset F$.

We say that a subset F of $\mathbb{N}_0$ is an **central*-set** if for every central set H in $\mathbb{N}_0$, $F \cap H \neq \emptyset$.

Distal point

Theorem (Furstenberg 1981)

Let (X,T) be a topological dynamical system and $x \in X$. Then the following assertions are equivalent:

1. *x is a distal point;*
2. *for every neighborhood U of x, the return time set $N(x,U)$ is a central*-set;*
3. *for every dynamical system (Y,S) and every almost periodic point $y \in Y$, (x,y) is almost periodic in the product system $(X \times Y, T \times S)$.*

Distal systems

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Theorem (L.-Yang 2024)

Let (X,T) be a minimal system. Then the following assertions are equivalent:

1. (X,T) *is distal;*
2. (X,T) *is* **pairwise IP*-equicontinuous** *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in X$ with $d(x,y) < \delta$, one has $\{i \in \mathbb{N} : d(T^i x, T^i y) < \varepsilon\}$ is an IP*-set;*
3. (X,T) *is* **pairwise central*-equicontinuous** *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in X$ with $d(x,y) < \delta$, one has $\{i \in \mathbb{N} : d(T^i x, T^i y) < \varepsilon\}$ is an central*-set.*

Li, Jian; Yang, Yini. Characterizations of distality via weak equicontinuity. Discrete Contin. Dyn. Syst. 44 (2024), no. 1, 61–77.

Product recurrence

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Theorem (Auslander-Furstenberg 1994)

Let (X,T) *be a* $\mathbb{Z}$*-action and* $x \in X$*. Then* x *is* $\mathbb{Z}$*-distal if and only if it is* $\mathbb{N}$*-distal.*

Question (Auslander-Furstenberg 1994)

1. How to characterize the closed subsemigroups S of a compact right topological semigroup for which a product S-recurrent point is a distal point?
   (When an $\mathcal{F}$-product recurrent point is distal?)
2. If (x, y) is recurrent for all almost periodic points y, is x necessarily a distal point?

Auslander, J.; Furstenberg, H. Product recurrence and distal points. Trans. Amer. Math. Soc. 343 (1994), no. 1, 221–232.

Definition

Let (X, T) be a dynamical system. A point $x \in X$ is said to be **weakly product recurrent** if for any dynamical system (Y, S) and any almost periodic point $y \in Y$, the point (x, y) is recurrent in the product system $(X \times Y, T \times S)$.

Theorem (Haddad-Ott 2008)

In the full shift $(\{0,1\}^{\mathbb{N}_0}, \sigma)$, every transitive point x is weakly product recurrent but not distal. So the second question of Auslander-Furstenberg (1994) is negative.

Haddad, Kamel; Ott, William. Recurrence in pairs. Ergodic Theory Dynam. Systems 28 (2008), no. 4, 1135–1143.

Furstenberg family

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Definition

Let $\mathcal{F}$ be a collection of subsets of $\mathbb{N}_0$. If $\mathcal{F}$ is hereditary upwards, that is, $F \in \mathcal{F}$ and $F \subset H \subset \mathbb{N}_0$ imply $H \in \mathcal{F}$, then we say that $\mathcal{F}$ is a **Furstenberg family**.

Definition

Let (X, T) be a topological dynamical system and $\mathcal{F}$ be a Furstenberg family. A point $x \in X$ is said to be $\mathcal{F}$-**recurrent** if for every neighborhood U of x,

$$N(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\} \in \mathcal{F}.$$

Product recurrence

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Definition

Let (X, T) be a topological dynamical system and $\mathcal{F}$ be a Furstenberg family. A point $x \in X$ is said to be $\mathcal{F}$-**product recurrent** if for any topological dynamical system (Y, S) and any $\mathcal{F}$-recurrent point $y \in Y$, the point (x, y) is recurrent in the product system $(X \times Y, T \times S)$.

Dong, Pandeng; Shao, Song; Ye, Xiangdong. Product recurrent properties, disjointness and weak disjointness. Israel J. Math. 188 (2012), 463–507.

Definition

We say that a subset F of $\mathbb{N}_0$ is **thick** if for any $N \in \mathbb{N}$ there exists $n \in \mathbb{N}_0$ such that $[n, n+N] \cap \mathbb{N}_0 \subset F$, and **piecewise syndetic** if it is an intersection of a syndetic set and a thick set.

Denote by $\mathcal{F}_{ps}$ the collection of piecewise syndetic subsets of $\mathbb{N}_0$.

Theorem (Huang-Ye 2005)

Let (X,T) be a topological dynamical system and $x \in X$. Then x is $\mathcal{F}_{ps}$-recurrent if and only if x is recurrent and the orbit closure of x has a dense set of almost periodic points.

$\mathcal{F}_{ps}$-product recurrence

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Theorem (Dong-Shao-Ye 2012)

If a subset F of $\mathbb{N}_0$ is thick, then there exists an $\mathcal{F}_{ps}$-recurrent point x in the full shift $(\{0,1\}^{\mathbb{N}_0}, \sigma)$ such that $N(x,[1]) \subset F \cup \{0\}$.

Theorem (Dong-Shao-Ye 2012)

For $\mathbb{N}_0$-system, every $\mathcal{F}_{ps}$-product recurrent point is almost periodic.

Question (Dong-Shao-Ye 2012)

Dose every $\mathcal{F}_{ps}$-product recurrent point is distal?

$\mathcal{F}_{ps}$-product recurrence

Theorem (Oproach-Zhang 2013)

If a subset F of $\mathbb{N}_0$ is central, then there exists an $\mathcal{F}_{ps}$-recurrent point x in the full shift $(\{0,1\}^{\mathbb{N}_0}, \sigma)$ such that $N(x,[1]) \subset F \cup \{0\}$.

Theorem (Oproach-Zhang 2013)

Every $\mathcal{F}_{ps}$-product recurrent point is distal.

Oprocha, Piotr; Zhang, Guohua. On weak product recurrence and synchronization of return times. Adv. Math. 244 (2013), 395–412.

Quasi-central set

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Theorem ((1)⇔(2) by Hindman-Maleki-Strauss 1996; (1)⇔(3) by Burns-Hindman 2007)

For a subset F of $\mathbb{N}_0$, the following assertions are equivalent:

(1) *F is quasi-central, that is, there exists an idempotent $p \in \beta\mathbb{N}_0$ such that $p \subset \mathcal{F}_{ps}$ and $F \in p$;*

(2) *there exists a decreasing sequence $\{F_n\}$ of piecewise syndetic subsets of F such that for any $n \in \mathbb{N}$ and $k \in F_n$ there exists $m \in \mathbb{N}$ such that $k + F_m \subset F_n$;*

(3) *there exists a topological dynamical system (X, T), $x \in X$, an $\mathcal{F}_{ps}$-recurrent point $y \in X$ and a neighborhood U of y such that for every neighborhood V of y, $N(x, V) \cap N(y, V)$ is piecewise syndetic and $N(x, U) \subset F \cup \{0\}$.*

Quasi-central set

- Hindman, Neil; Maleki, Amir; Strauss, Dona. Central sets and their combinatorial characterization. J. Combin. Theory Ser. A 74 (1996), no. 2, 188–208.
- Burns, Shea D.; Hindman, Neil. Quasi-central sets and their dynamical characterization. Topology Proc. 31 (2007), no. 2, 445–455.
- Li, Jian. Dynamical characterization of C-sets and its application. Fund. Math. 216 (2012), no. 3, 259–286.

Return time sets

Theorem (L.-Liang-Yang, preprint)

1. *Given a topological dynamical system* (X,T)*, if* $x \in X$ *is an* $\mathcal{F}_{ps}$*-recurrent point, then for every neighborhood* U *of* x*,* $N(x,U)$ *is a quasi-central set.*
2. *For a quasi-central subset* F *of* $\mathbb{N}_0$*, there exists a topological dynamical system* (X,T) *and an* $\mathcal{F}_{ps}$*-recurrent point* $x \in X$ *and a neighborhood* U *of* x *such that* $N(x,U) \subset F \cup \{0\}$*.*

Note: This result holds for countable infinite discrete group actions.

Product recurrence

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Theorem (L.-Liang-Yang, preprint)

Let G be a countable infinite discrete group. If a closed subsemigroup S of βG contains the smallest ideal $\mathcal{K}(\beta G)$ of βG, then for any Ellis action $\Phi\colon \beta G \times X \to X$, a point $x \in X$ is distal if and only if it is S-product recurrent.

Note: This partly answers the first question of Auslander-Furstenberg (1994).

Product recurrence

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Theorem (L.-Liang-Yang, preprint)

Let G be a countable infinite discrete group. If $\mathcal{F}$ is $\mathcal{F}_{ps}$, $\mathcal{F}_{inf}$ (the collection of infinite subsets of G), or the collection of subsets of G with positive upper Banach density when G is amenable, then for any G-system (X, G) and $x \in X$, the following assertions are equivalent:

1. *$x \in X$ is distal;*
2. *x is $\mathcal{F}$-product recurrent;*
3. *for any G-system (Y, G) and any $\mathcal{F}$-recurrent point $y \in Y$, the point (x, y) is $\mathcal{F}$-recurrent in the product system $(X \times Y, G)$.*

Note: This generalized Oproach-Zhang's result to group actions.

Group action

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Definition

Let X be a compact Hausdorff space and G be a discrete group with an identity e. If $\Pi : G \times X \to X$ is a continuous map and satisfies

1. $\Pi(e, x) = x$, $x \in X$;
2. $\Pi(s, \Pi(g, x)) = \Pi(sg, x)$, $x \in X, s, g \in G$,

then the triple (X, G, Π) is called a G-**system**.

We will briefly say that the tuple (X, G) as a G-system and $gx := \Pi(g, x)$ if there is no ambiguity.

Key ingredient

Definition

We say that a Furstenberg family $\mathcal{F}$ satisfies **(P1)** if for any $A \in \mathcal{F}$ there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of G such that

1. for every $n \in \mathbb{N}$, $A_n \subset A$;
2. for every $m, n \in \mathbb{N}$ with $n \neq m$, $A_n \cap A_m = \emptyset$;
3. for every strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, $\bigcup_{k=1}^{\infty} A_{n_k} \in \mathcal{F}$.

Key ingredient

Definition

We say that a Furstenberg family $\mathcal{F}$ satisfies **(P2)** if for any $A \in \mathcal{F}$ and any finite subset K of G there exists a subset B of A such that $B \in \mathcal{F}$ and for any distinct $k_1, k_2 \in K \cup \{e\}$, $k_1 B \cap k_2 B = \emptyset$.

Lemma

Let $\mathcal{F}$ be a proper Furstenberg family. If $\mathcal{F}$ satisfies the following conditions:

1. *for every $A \in \mathcal{F}$ and $g \in G$, $gA \in \mathcal{F}$;*
2. *$\mathcal{F}$ has the Ramsey property, that is, $F_1 \cup F_2 \in \mathcal{F}$ implies either $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$,*

then $\mathcal{F}$ satisfies (P2).

Key ingredient

Lemma

Let G be a countable infinite group. $\mathcal{F}_{ps}$ and $\mathcal{F}_{inf}$ satisfy (P1) and (P2).

If G is amenable, then the collection of subsets of G with positive upper Banach density satisfies (P1) and (P2).

Key ingredient

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Theorem

Let G be a countable infinite group and $\mathcal{F}$ be a Furstenberg family satisfying (P1) and (P2). For any $F \in \mathcal{F}$, the following assertions are equivalent:

1. *there exists a compact metric G-system (X, G), an $\mathcal{F}$-recurrent point $x \in X$ and a neighborhood U of x such that $N(x, U) \subset F \cup \{e\}$;*
2. *there exists an $\mathcal{F}$-recurrent point x in the full shift $(\{0, 1\}^G, G)$ such that $N(x, [1]) \subset F \cup \{e\}$;*
3. *there exists a decreasing sequence $\{F_n\}$ of F with $F_n \in \mathcal{F}$ such that for any $n \in \mathbb{N}$ and $f \in F_n$ there exists $m \in \mathbb{N}$ such that $fF_m \subset F_n$.*

Note: this proof is "purely" combinatorial.

Key ingredient

Using the ultrafilter representation of the Stone-Čech compactification βG of G and algebra properties of βG, one can extend the above result to G-system on compact Hausdorff spaces with the condition $h(\mathcal{F})$ is a non-empty closed subsemigroup of βG.

Thanks for your attention!