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On the entropy paradox on tame graphs

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Entropy paradox

For given space X, the family of exact maps $\mathcal{E}(X)$ attains lower entropy than the family of pure mixing maps $\mathcal{PM}(X)$, meaning that

$$\inf\{h(f)\colon f \in \mathcal{E}(X)\} < \inf\{h(g)\colon g \in \mathcal{PM}(X)\}.$$

Interval maps

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Theorem (Harańczyk, Kwietniak)

For transitive maps on the unit interval $I = [0,1]$, *while the infimum* $\inf\{h(g)\colon g \in \mathcal{PM}(I)\} = \frac{\log 3}{2}$, *we have that the infimum* $\inf\{h(f)\colon f \in \mathcal{E}(I)\} = \frac{\log 2}{2}$.

Pure mixing maps from the exact ones

Having a suitable piecewise linear exact map, use rescaling and overlapping to make a related pure mixing one.

Theorem (Harańczyk, Kwietniak, Oprocha)

Let G be a topological graph. Then we have that the infimum $\inf\{h(f)\colon f \in \mathcal{PM}(G)\} = \frac{\log 3}{\Lambda(G)} > 0$, *while in certain cases exact maps can attain arbitrarily low entropy.*

Tree maps

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Theorem (Harańczyk, Kwietniak, Oprocha)

Let T_n be a full binary tree of height $n \in \mathbb{N}$. Then for any $\epsilon > 0$ there exists a piecewise linear exact map $f : T_n \to T_n$ such that the root of T_n is a fixed point for f, the endpoints of T_n form a sinle periodic orbit and the topological entropy $h(f)$ satisfies

$$\frac{\log 3}{2^n} < h(f) < \frac{\log 3}{2^n} + \epsilon.$$

Intermission

Bounding the entropy

Let $f\colon X \to X$ be an L-Lipschitz map. Then the topological entropy $h(f) \leq \log L \cdot \dim_H(X)$.

Theorem (Alseda, Misiurewicz)

If $f\colon G \to G$ is a continuous piecewise monotone transitive graph map and $h(f) = \log \beta > 0$, then f is conjugate to some continuous piecewise monotone graph map $g\colon G \to G$ with constant slope β.

The dendrite case

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Theorem (Spitalsky)

Let X be a dendrite with infinitely many free arcs. Then we have that

$$\inf\{h(f)\colon f \in \mathcal{E}(X)\} = 0.$$

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Theorem (Kwietniak, Oprocha, T)

Let $\mathcal{G}$ be the Gehman dendrite. Then for any $\epsilon > 0$ there exists a pure mixing map $f : \mathcal{G} \to \mathcal{G}$ such that its topological entropy $h(f) < \epsilon$.

Tame graphs

Definition

We say that a one dimensional continuum is a tame graph if the closure of the set of its endpoints and branching points is countable.

Theorem

There exists a pure mixing map on a tame graph with arbitrarily low entropy.

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