The Appeal of Pointfree Topology (to classical topologists): *♪♪ These are a few of my favorite things ♪♪*

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The Appeal of Pointfree Topology

These are a few of my favorite things:

1. Isbell's density theorem
2. Cozero "tree rings" of a completely regular frame
3. Scaffolding of a frame
4. Paracompactness
5. Bruns-Lakser completion of a meet-semilattice
6. Coz inclusions
7. … paddling with dolphins.

Some basic concepts in frame theory

Categories: Frm and Loc **Top**

A *frame* is a complete lattice L which satisfies Opens of $X \in \mathbf{Top}$

$$x \wedge \bigvee x_i = \bigvee (x \wedge x_i) \text{ for all } x, x_i \in L$$

A *frame map* preserves finite meets and arbitrary joins. preimage of continuous maps

$$\mathbf{Loc} = \mathbf{Frm}^{op}$$

Examples: Complete Boolean algebras

For $X \in \mathbf{Top}$, open sets $\mathcal{O}X$ of X

Sublocales/quotient frames/nuclei/congruences

Every frame is a complete Heyting algebra: $x \to y = \bigvee\{z : z \wedge x \leq y\}$

Note that $z \wedge x \leq y$ iff $z \leq x \to y$ Galois connection

A subset S of L is a sublocale if subspace

1. S is closed under $\bigwedge$
2. $x \to s \in S$ for all $s \in S$ and $x \in L$.

- S has the same meets as L
- $1 \in S$
- joins in S may be different from joins in L

The collection of all sublocales form a coframe: (arbitrary) meet is intersection.

Sublocales/quotient frames/nuclei/congruences

A frame map $h : L \longrightarrow M$ has a right adjoint:

- $h_*(x) = \bigvee\{a \in L \mid h(a) = x\}$ largest element mapped to x
- $h(h_*(x)) = x$

If h is onto, $h_*[M] \cong M$,

- $h_*[M]$ is a sublocale of L

$h_* \circ h$ is a nucleus

If S is a sublocale of L, define a frame map $h : L \longrightarrow S$

$$h(a) = \bigwedge\{s \in S \mid s \geq a\}$$

- h is an onto i.e. a frame quotient

Open sublocale vs. open quotient

In general, an element a in L can be “removed” by collapsing the frame:

- frame quotient

$$L \twoheadrightarrow {\downarrow} a$$

$$b \mapsto a \wedge b$$

- corresponding sublocale: each b in L is mapped to $a \to b$

$$\mathfrak{o}(a) = \{a \to b : b \in L\}$$

“subspace induced by an open set”

How to remove a point?

Consider $X \smallsetminus \{x\}$: Topological inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is represented by $\mathfrak{o}(a)$ where $a = X \smallsetminus \{x\}$

Skeleton a.k.a. Booleanization $\mathfrak{B}L$

a^* denotes the *pseudocomplement* of $a \in L$: "complement of closure"

- largest element disjoint from a
- $a^* = \bigvee\{x | x \wedge a = 0\} = a \to 0$
- $a^* = a^{***}$

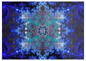
a is *dense* if $a^* = 0$ "closure" is the top

a^{**} denotes the *double pseudocomplement* "interior of closure"

$\mathfrak{B}L = \{a^{**} : a \in L\}$ is the Booleanization/skeleton "regular opens"

- complete Boolean algebra
- sublocale of L
- has no proper dense elements

1 Isbell's Density Theorem



Dense sublocales/quotient frames

S is a dense sublocale of L if $0 \in S$

"closure" of S is L

- the frame morphism maps only 0 to 0;
- $0 = 0^{**}$ so $0 \in \mathfrak{B}L$

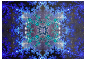
Theorem (Isbell)

Every locale has a smallest dense sublocale namely $\mathfrak{B}L = \{x^{**} : x \in L\}$.

S is dense iff $\mathfrak{B}L \subseteq S$

No spatial analog

The skeleton of the (natural) topology on the Reals



$\mathfrak{B}\mathscr{O}\mathbb{R}$, the skeleton of the reals: all regular opens (a.k.a. open domains)

- every open interval is a regular open
- union of two "separated" open intervals where (p,q) is "separated" from (r,s) if $q<r$ "not touching"

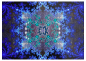
Each real number r corresponds to the open set $(-\infty,r)\cup(r,\infty)$

- not regular opens
- dense in $\mathscr{O}\mathbb{R}$
- $\mathfrak{o}((-\infty,r)\cup(r,\infty))$ is a dense sublocale of $\mathscr{O}\mathbb{R}$ skeleton has no points

$$\mathfrak{B}\mathscr{O}\mathbb{R}\subseteq\bigcap_{r\in\mathbb{R}}\mathfrak{o}((-\infty,r)\cup(r,\infty))$$

Pointfree view of the rationals $\mathbb{Q}$ and irrationals $\mathbb{P}$

$$\mathbb{R} = \mathbb{Q} \,\dot{\cup}\, \mathbb{P}$$



$\mathbb{Q} = \mathbb{R} \smallsetminus \mathbb{P}$

$\mathbb{P} = \mathbb{R} \smallsetminus \mathbb{Q}$

Sublocale of the rationals: $\mathscr{O}\mathbb{Q}$

Sublocale of the irrationals: $\mathscr{O}\mathbb{P}$

Remove all the irrational points:

Remove all the rational points: $\mathscr{O}\mathbb{Q} = \bigcap_{r \in \mathbb{P}} \mathfrak{o}((-\infty, r) \cup (r, \infty))$

$\mathscr{O}\mathbb{P} = \bigcap_{r \in \mathbb{Q}} \mathfrak{o}((-\infty, r) \cup (r, \infty))$

Dense sublocale of $\mathscr{O}\mathbb{R}$

Dense sublocale of $\mathscr{O}\mathbb{R}$

$$\mathfrak{B}\mathscr{O}\mathbb{R} \subseteq \bigcap_{r \in \mathbb{R}} \mathfrak{o}((-\infty, r) \cup (r, \infty)) = \mathscr{O}\mathbb{Q} \cap \mathscr{O}\mathbb{P}$$

Some other special elements of a frame

Rather below relation: $b \prec a$ means that $b^* \vee a = 1$

$\mathrm{Coz}\, L$: **cozero elements**

- $a \in L$ is cozero if $a = \bigvee_n \{a_n | a_n \prec a\}$ "cozero subset"
- largest regular sub-σ-frame of L

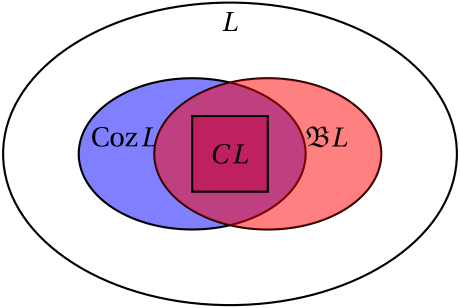
A frame L is *completely regular* if it is join-generated by $\mathrm{Coz}\, L$

CL: **complemented elements**

- $a \in L$ is complemented if $a \prec a$ "clopen subset"
- boolean sub-lattice of L

A frame L is *zero-dimensional* if it is join-generated by CL

Cozeros vs. Skeleton



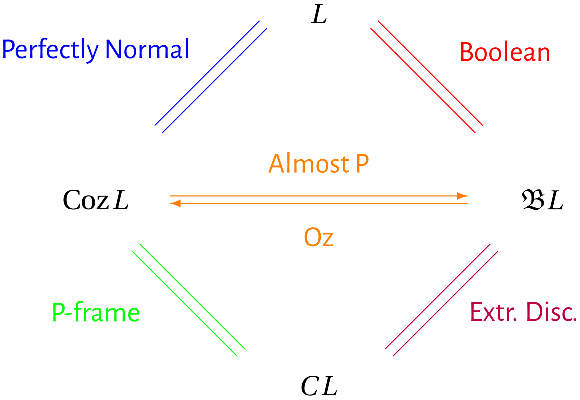
perfectly normal iff $L = \mathrm{Coz}\, L$ oz iff $\mathfrak{B}L \subseteq \mathrm{Coz}\, L$

ω_1-hollow (almost P) iff $\mathrm{Coz}\, L \subseteq \mathfrak{B}L$ P-frame iff $\mathrm{Coz}\, L = CL$

extremally disconnected iff $\mathfrak{B}L = CL$

almost Boolean iff $\mathrm{Coz}\, L = \mathfrak{B}L$ $(= CL)$

Summary of $\operatorname{Coz} L$ vs $\mathfrak{B}L$



2 The tree rings of a (completely regular) frame determined by Coz

Kappa cozeros: Perfectly normal degree

$\operatorname{Coz}_\kappa L$: **$\kappa$-joins of cozeros**

- completely regular κ-frames
- Ascending sequence of completely regular sub-κ-frames of L: Cozero tower

$$\operatorname{Coz} L \subseteq \cdots \subseteq \operatorname{Coz}_\kappa L \subseteq \cdots \subseteq \operatorname{Coz}_\rho L = L$$

The least such cardinal is called the perfectly normal degree of L

- L is perfectly normal iff PN degree is ω_1 for example, $\mathcal{O}\mathbb{R}$

Lindelöf coreflection

Theorem (Madden & Vermeer)

Lindelöf completely regular frames are a coreflective (full) subcategory of completely regular frames

No spatial analogue!

- denoted by $\mathscr{L}_{\omega_1} L$: all ω_1-ideals of $\mathrm{Coz}\, L$ (downsets closed under countable joins)
- "free" frame over $\mathrm{Coz}\, L$ (as a ω_1-frame)
- coreflection map is given by join and is a frame quotient, so L may be identified as a sublocale of $\mathscr{L}_{\omega_1} L$
- $\mathrm{Coz}\, L = \mathrm{Coz}\, \mathscr{L}_{\omega_1} L$

κ-Lindelöf coreflections

Theorem (Madden & Vermeer)

κ-Lindelöf completely regular frames are a coreflective (full) subcategory of completely regular frames

- denoted by $\mathscr{L}_\kappa L$: all κ-ideals of $\mathrm{Coz}_\kappa L$ (downsets closed under κ-joins)
- "free" frame over $\mathrm{Coz}_\kappa L$ (as a κ-frame)
- coreflection map is given by join and is a frame quotient, so L may be identified as a sublocale of $\mathscr{L}_\kappa L$
- $\mathrm{Coz}_\kappa L = \mathrm{Coz}_\kappa \mathscr{L}_\kappa L$

The "Lindelöf" tower

- $\operatorname{Coz} L = \operatorname{Coz} \mathscr{L}_\kappa L$ for all κ
- $\operatorname{Coz}_\kappa L = \operatorname{Coz}_\kappa \mathscr{L}_\gamma L$ for all $\gamma \geq \kappa$

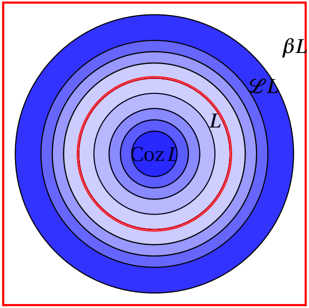
Ascending sequence of sublocales: Lindelöf tower

$$L = \mathscr{L}_\mu L \subseteq \cdots \subseteq \mathscr{L}_\gamma L \subseteq \cdots \subseteq \mathscr{L}_\kappa L \subseteq \cdots \subseteq \mathscr{L}_{\omega_1} L$$

where $\mu \geq \gamma \geq \kappa \geq \omega_1$
The least such cardinal is called the Lindelöf degree of L

- L is Lindelöf iff the Lindelöf degree $\mu \leq \omega_1$ For example, the Lindelöf degree of $\mathscr{O}\mathbb{R}$ is ω_1

Building the cozero "tree rings"

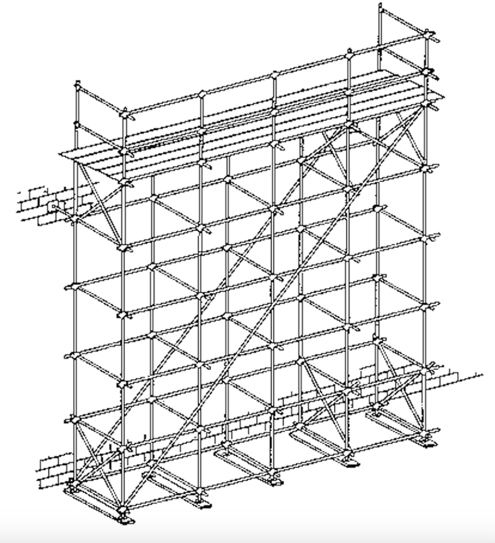


$$\operatorname{Coz} L \subseteq L \subseteq \mathscr{L}L$$

$$\operatorname{Coz} L \subseteq \cdots \subseteq \operatorname{Coz}_\kappa L \subseteq \cdots \subseteq \operatorname{Coz}_\rho L = L$$

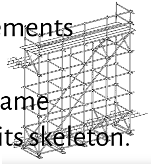
$$L = \mathscr{L}_\mu L \subseteq \cdots \subseteq \mathscr{L}_\gamma L \subseteq \cdots \subseteq \mathscr{L}_\kappa L \subseteq \cdots \subseteq \mathscr{L}_{\omega_1} L$$

3 The Scaffolding of a completely regular frame



Hollowing out of a frame

A frame is λ-hollow if it has no proper dense λ-cozero elements



- λ-hollow $\Leftrightarrow \mathrm{Coz}_\lambda L \subseteq \mathfrak{B}L$
- ω_1-hollow $\Leftrightarrow \mathrm{Coz}\, L \subseteq \mathfrak{B}L$, that is L is an almost P-frame
- hollow (if λ- hollow for all λ) $\Leftrightarrow L = \mathfrak{B}L$, that is L is its skeleton.

Theorem (Ball, Hager, WW)

λ-hollow completely regular frames are a reflective (non-full) subcategory of completely regular frames

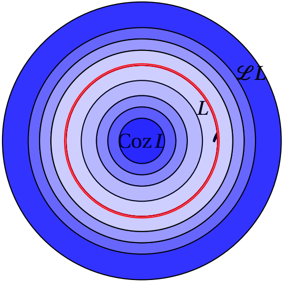
- denoted by $\mathscr{H}_\lambda L$
- intersection of all the dense λ-cozero sublocales of L remove all dense λ-cozeros

$$\mathscr{H}_\lambda L = \bigcap_{\text{dense}\, a \in \mathrm{Coz}_\lambda L} \mathfrak{o}(a)$$

Hollowing sequence (descending):

$$L \supseteq \mathscr{H}_{\omega_1} L \supseteq \cdots \supseteq \mathscr{H}_\lambda L \subseteq \cdots \supseteq \mathscr{H}_\rho L = \mathfrak{B}L$$

Building the scaffolding of a completely regular frame L



$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1}L \arrow[dd] \\
Coz_{\omega_1}L \arrow[ur, hook] \arrow[dd, hook] & \\
 & \mathscr{L}_{\omega_2}L \arrow[dd, dashed] \\
Coz_{\omega_2}L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \arrow[dd, dashed] \\
Coz_{\mu}L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \\
Coz_{\rho}L = L \arrow[ur, hook] &
\end{tikzcd}
$$

Building the scaffolding of a completely regular frame L

$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1} L \arrow[dd] \\
Coz_{\omega_1} L \arrow[ur, hook] \arrow[dd, hook] & \\
 & \mathscr{L}_{\omega_2} L \arrow[dd, dashed] \\
Coz_{\omega_2} L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \arrow[dd, dashed] \\
Coz_{\mu} L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \\
Coz_{\rho} L = L \arrow[ur, hook] &
\end{tikzcd}
$$

The scaffolding of a completely regular frame L

$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1}L \arrow[r] \arrow[dd] & \mathscr{H}_{\omega_1}\mathscr{L}_{\omega_1}L \\
Coz_{\omega_1}L \arrow[ur, hook] \arrow[dd, hook] & & \\
 & \mathscr{L}_{\omega_2}L \arrow[r] \arrow[dd, dashed] & \mathscr{H}_{\omega_2}\mathscr{L}_{\omega_2}L \\
Coz_{\omega_2}L \arrow[ur, hook] \arrow[dd, hook, dotted] & & \\
 & L \arrow[r] \arrow[dd, dashed] & \mathscr{H}_{\mu}L \\
Coz_{\mu}L \arrow[ur, hook] \arrow[dd, hook, dotted] & & \\
 & L \arrow[r] & \mathfrak{B}L \\
Coz_{\rho}L = L \arrow[ur, hook] & &
\end{tikzcd}
$$

The scaffolding of a completely regular frame L

$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1}L \arrow[r] \arrow[dd] & \mathscr{H}_{\omega_1}\mathscr{L}_{\omega_1}L \arrow[dd] \\
Coz_{\omega_1}L \arrow[ur, hook] \arrow[dd, hook] & & \\
 & \mathscr{L}_{\omega_2}L \arrow[r] \arrow[dd, dashed] & \mathscr{H}_{\omega_2}\mathscr{L}_{\omega_2}L \arrow[dd, dashed] \\
Coz_{\omega_2}L \arrow[ur, hook] \arrow[dd, hook, dotted] & & \\
 & L \arrow[r] \arrow[dd, dashed] & \mathscr{H}_{\mu}L \arrow[dd, dashed] \\
Coz_{\mu}L \arrow[ur, hook] \arrow[dd, hook, dotted] & & \\
 & L \arrow[r] & \mathfrak{B}L \\
Coz_{\rho}L = L \arrow[ur, hook] & &
\end{tikzcd}
$$

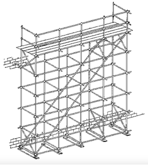
The scaffolding of a completely regular frame L

$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1}L \arrow[rr] \arrow[dd] & & \mathscr{H}_{\omega_1}\mathscr{L}_{\omega_1}L \arrow[dd] \\
Coz_{\omega_1}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook] & & \mathfrak{B}_{\omega_1}L \arrow[dd, hook] & \\
 & \mathscr{L}_{\omega_2}L \arrow[rr] \arrow[dd, dashed] & & \mathscr{H}_{\omega_2}\mathscr{L}_{\omega_2}L \arrow[dd, dashed] \\
Coz_{\omega_2}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook, dotted] & & \mathfrak{B}_{\omega_2}L \arrow[dd, hook, dotted] & \\
 & L \arrow[rr] \arrow[dd, dashed] & & \mathscr{H}_{\mu}L \arrow[dd, dashed] \\
Coz_{\mu}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook, dotted] & & \mathfrak{B}_{\mu}L \arrow[dd, hook, dotted] & \\
 & L \arrow[rr] & & \mathfrak{B}L \\
Coz_{\rho}L = L \arrow[ur, hook] \arrow[rr, two heads] & & \mathfrak{B}L &
\end{tikzcd}
$$

The scaffolding of a completely regular frame L

$$
\begin{tikzcd}
 & \mathscr{L}_{\omega_1}L \arrow[rr] \arrow[dd] & & \mathscr{H}_{\omega_1}\mathscr{L}_{\omega_1}L \arrow[dd] \\
Coz_{\omega_1}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook] & & \mathfrak{B}_{\omega_1}L \arrow[ur, hook] \arrow[dd, hook] & \\
 & \mathscr{L}_{\omega_2}L \arrow[rr] \arrow[dd, dashed] & & \mathscr{H}_{\omega_2}\mathscr{L}_{\omega_2}L \arrow[dd, dashed] \\
Coz_{\omega_2}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook, dotted] & & \mathfrak{B}_{\omega_2}L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \arrow[rr] \arrow[dd, dashed] & & \mathscr{H}_{\mu}L \arrow[dd, dashed] \\
Coz_{\mu}L \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, hook, dotted] & & \mathfrak{B}_{\mu}L \arrow[ur, hook] \arrow[dd, hook, dotted] & \\
 & L \arrow[rr] & & \mathfrak{B}L \\
Coz_{\rho}L = L \arrow[ur, hook] \arrow[rr, two heads] & & \mathfrak{B}L \arrow[ur, hook] &
\end{tikzcd}
$$

The scaffolding of the reals

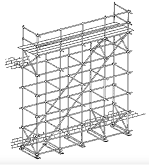


Recall:

- $\mathscr{O}\mathbb{R}$ is perfectly normal so $\mathscr{O}\mathbb{R} = \mathrm{Coz}\,\mathscr{O}\mathbb{R}$
- $\mathfrak{B}\mathscr{O}\mathbb{R} = \mathfrak{B}_\kappa\mathscr{O}\mathbb{R}$ for all $\kappa \geq \omega_1$
- $\mathscr{O}\mathbb{R}$ is Lindelöf so $\mathscr{O}\mathbb{R} = \mathscr{L}_\kappa\mathscr{O}\mathbb{R}$ for all $\kappa \geq \omega_1$

$$\mathscr{O}\mathbb{R} \longrightarrow \mathfrak{B}\mathscr{O}\mathbb{R}$$

The scaffolding of $\beta\mathbb{R}$



Recall:

- $\mathscr{O}\beta\mathbb{R}$ is compact, hence Lindelöf
- $\mathscr{O}\beta\mathbb{R}$ is NOT perfectly normal so $\mathrm{Coz}\,\mathscr{O}\beta\mathbb{R} \subset \mathscr{O}\beta\mathbb{R}$
- but, $\mathfrak{B}\mathscr{O}\mathbb{R} \cong \mathfrak{B}\mathscr{O}\beta\mathbb{R} = \mathfrak{B}_\kappa\mathscr{O}\beta\mathbb{R}$ for all $\kappa \geq \omega_1$

$$\begin{tikzcd}
Coz_{\omega_1}\beta\mathbb{R} \arrow[d, hook] \arrow[dr, two heads] & \\
Coz_{\omega_2}\beta\mathbb{R} \arrow[d, hook, dotted] \arrow[r, two heads] & \mathfrak{B}\mathscr{O}\mathbb{R} \\
\mathscr{O}\beta\mathbb{R} \arrow[ur, two heads] &
\end{tikzcd}$$

Some other favorites...

4 Paracompactness

- Coreflective subcategory of completely regular frames
- No spatial analog
- Uses the completion of the fine uniformity...

5 Bruns-Lakser completion of a meet-semilattice

- injective hull of the meet-semilattice
- always a frame (cf. Dedekind-McNeille completion)

6 Coz inclusions

- use interaction of cozeros and sublocales to understand weakenings of C-embeddings that are different from C^*-embeddings, but are also stronger than Z-embeddings.

Advertising Moment: Banaschewski 100

CHAPMAN UNIVERSITY

Obrigada :)

https://sites.google.com/chapman.edu/banaschewski100/