Permutation Models Arising from Topological Spaces

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What is a Permutation Model?

Definition

A dynamical ideal is a triple $(\Gamma \curvearrowright X, I)$ where Γ is a group acting on the set X and I is an ideal invariant under the group action.

A dynamical ideal can be used to construct a permutation model of set theory:

Definition

Given a dynamical ideal $(\Gamma \curvearrowright X, I)$, let $V[[X]]$ denote a model of *ZFCA* using X as the set of atoms. The permutation model associated to the dynamical ideal is the transitive part of $\{A \in V[[X]] : \exists b \in I : \mathrm{pstab}(b) \subseteq \mathrm{stab}(A)\}$.

σ-Complete Ideals

Definition

Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. The dynamical ideal is σ-complete if for every set $a \in I$ and every countable sequence $(b_n : n \in \omega) \subseteq I$ there are group elements $\gamma_n \in \mathrm{pstab}(a)$ such that $\bigcup_n \gamma_n \cdot b_n \in I$.

Theorem

(Zapletal, 2023) Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. If the dynamical ideal is σ-complete, then the associated permutation model satisfies the axiom of countable choice.

A Tool for σ-completeness

Definition

Let $(Homeo(X) \curvearrowright X, I)$ be a dynamical ideal, and let d be a compatible metric on X. We say the dynamical ideal is tight if for all $a, b \in I$ and for all $\varepsilon > 0$, there is $\gamma \in pstab(a)$ such that $\gamma \cdot b \subseteq Ball(a, \varepsilon)$.

Proposition

(Y.) Let I be the ideal of countable closed sets on a metrizable space X. If the dynamical ideal $(Homeo(X) \curvearrowright X, I)$ is tight, then it is σ-complete.

Proof.

Choose γ_n such that $\gamma_n \cdot b_n \subseteq Ball(a, 1/n)$ and use sequential closure to show that $a \cup \bigcup \gamma_n \cdot b_n$ is closed. □

Examples of Tight Ideals

Example

(Y.) Let $X = 2^\omega$ and I the ideal generated by countable closed sets. Then $(Homeo(X) \curvearrowright X, I)$ is tight.

Proof.

1. Let a, b, ε be given. Let $\mathcal{C}$ be a finite cover of a by pairwise disjoint balls of radius $< \varepsilon$.
2. Let $\mathcal{D}$ be a finite cover of b by disjoint balls such that $\bigcup \mathcal{D}$ is disjoint from $\bigcup \mathcal{C}$.
3. By induction on $\mathcal{D}$, move sets in $\mathcal{D}$ to $\bigcup \mathcal{C}$.

$\square$

Examples of Tight Ideals

Example

(Y.) Let $X = \mathbb{R}^n$ and I the ideal generated by countable compact sets. Then $(Homeo(X) \curvearrowright X, I)$ is tight.

Proof.

1. Let a, b, ε be given, and let $\mathcal{C}$ cover of a by pairwise disjoint balls of radius $< \varepsilon$ with boundaries disjoint from b.
2. Replace b with $b \setminus \bigcup \mathcal{C}$ and a with $a \cup (b \cap \bigcup \mathcal{C})$.
3. Let $\mathcal{D}$ cover b by pairwise disjoint balls with $\bigcup \mathcal{D} \cap \bigcup \mathcal{C} \neq \emptyset$.
4. Given $C \in \mathcal{C}$ and $D \in \mathcal{D}$, find a set K which is the image of the unit circle under a self-homeomorphism of $\mathbb{R}^n$, contains C and D and does not meet any other set in either cover.
5. Apply the Annulus Theorem to the region obtained in the previous step.

$\square$

Modifications

Example

(Y.) Let $X = \mathbb{R}^n$ and I the ideal generated by countable closed sets. Then $(Homeo(X) \curvearrowright X, I)$ is tight.

Proof.

Given $a, b \in I$, tile $\mathbb{R}^n$ with "cubes" such that the boundaries avoid a, b. Now deal with each cube individually. □

Example

(Y.) Let $X = [0,1]^n$ and I the ideal generated by countable closed sets. Then $(Homeo(X) \curvearrowright X, I)$ is tight.

Proof.

Use a sphere to separate the cube into two parts which can be dealt with individually. □

A Non-example

Proposition

Let (X, d) be a separable metric space, and let I contain all countable closed sets. Then $(Iso(X, d) \curvearrowright (X, d), I)$ is not σ-complete.

Proof.

1. Let $a = \emptyset$ and for each n let b_n be a $1/n$ net of X.
2. Note that regardless of the choice of γ_n, $\bigcup \gamma_n \cdot b_n$ will be a dense set.

□

Cofinal Orbits

Definition

Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. It has cofinal orbits if for every $a \in I$ there exists a $b \in I$ which is a-large: for every $c \in I$ there exists $\gamma \in pstab(a)$ such that $c \subseteq \gamma \cdot b$.

Theorem

(Zapletal, 2023) Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal with cofinal orbits. The corresponding permutation model satisfies the axiom of well-ordered choice.

An example of cofinal orbits

Example

(Y., independently discovered by M. Elekes) Let $X = [0,1]^n$, and I the ideal generated by closed nowhere dense sets. Then $(Homeo(X) \curvearrowright X, I)$ has cofinal orbits.

The argument involves building a Sierpiński carpet on top of nowhere dense sets:

Sierpiński Carpet

Definition

$A \subseteq X = [0,1]^n$ is an $n-1$-dimensional Sierpiński carpet filling X if

1. A is closed nowhere dense
2. $bd(X) \subseteq A$
3. The set of components of $X \setminus A$, $\{U_i : i \in \omega\}$ is such that $diam(U_i) \to 0$
4. $\{\overline{U_i} : i \in \omega\}$ is pairwise disjoint

The following is a corollary of theorems of Whyburn ($n = 1$, 1958) and Cannon ($n \geq 2$, 1972):

Lemma

Given Sierpiński carpets $A, B \subseteq X$, there is a self-homeomorphism of X such that $h(A) = B$ and $h|bd(X) = id$.

Sierpiński Carpet

Lemma

Given $A \subseteq [0,1]^n$, there is a Sierpiński Carpet B filling $[0,1]^n$ such that $A \subseteq B$.

Proof.

Fix a countable dense subset $\{x_i : i \in \omega\}$ and construct U_i centered at x_i such that

1. $diam(U_i) < 1/i$
2. $\overline{U_i}$ is disjoint from A and $\overline{U_j}$ for $j < i$.

In the end, let $B = [0,1]^n \setminus \bigcup U_i$. $\square$

NWD has Cofinal Orbits

We are finally ready to state the proof that $Homeo([0,1]^n \curvearrowright [0,1]^n, NWD)$ has cofinal orbits.

Proof.

1. Given $a \in I$, let $K \supseteq a$ be a Sierpiński carpet filling $[0,1]^n$.
2. For each complementary component U_i of $[0,1]^n \setminus K$, let b_i be a Sierpiński carpet filling $\overline{U_i}$. Let $b = K \cup \bigcup b_i$.
3. Given $c \in I$, to see b is a-large, apply the corollary from Whyburn and Cannon to each $\overline{U_i}$ to get γ_i which moves b_i onto the corresponding portion of c.
4. Finally, paste all of the γ_i together.

□

Modifications

Definition

Let I_C be an ideal on $[0,1]^n$ and define a corresponding ideals on $\mathbb{R}^n$:

$I_S = \{a :$ for any homeomorphic image of the cube $C, a \cap C \in I_C\}$

$I_{SB} = \{a : a \in I_S$ and a is bounded$\}$

Proposition

(Y.) If $bd[0,1]^n \in I_C$ and $(Homeo([0,1]^n) \curvearrowright [0,1]^n, I_C)$ has cofinal orbits, then so does $(Homeo(\mathbb{R}^n) \curvearrowright \mathbb{R}^n, I_S)$ as well as $(Homeo(\mathbb{R}^n) \curvearrowright \mathbb{R}^n, I_{SB})$.

Proof.

Tile the space with copies of the cube. $\square$

Further work

1. Does the result for the ideal of countable closed sets generalize to infinite-dimensional space?
2. Does the result for the ideal of closed nowhere dense sets generalize to infinite-dimensional space?
3. In $\mathbb{R}^2$, the ideal of compact 0-dimensional sets satisfies a criterion that implies the permutation model satisfies the axiom of dependent choice. Can the ideal be shown to have cofinal orbits?
4. Is there a uniform way to make connections between an ideal of closed sets and the corresponding ideal of compact sets?

Thank you!