Mixing composition operators and Kitai's criterion

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July 9, 2024

SUMTOPO
2024
University of Coimbra, Portugal

Joint work with Daniel Gomes (Campinas, Brazil)

Concepts in linear dynamics

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$$T : X \to X$$

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A vector $x \in X$ is called hypercyclic for T if its orbit

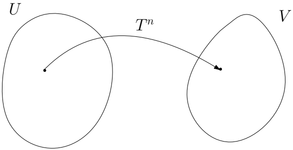
$$\text{orb}(x, T) = \{x, Tx, T^2x, \ldots\} \text{ is dense in } X.$$

Then T is called a hypercyclic operator.

Concepts in linear dynamics

By Birkhoff, T is hypercyclic if and only if T is topologically transitive, that is, for any non-empty open sets $U, V \subset X$, there is $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \varnothing.$$



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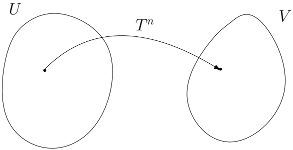

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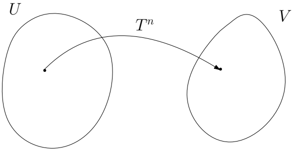

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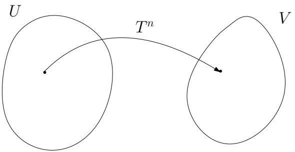

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It's equivalent to saying that each subsequence $(T^{n_k})_k$ admits a dense orbit.

Concepts in linear dynamics

The first sufficient condition for hypercyclicity:

Theorem (Kitai, thesis 1982)

Suppose that there are dense subsets $X_0, Y_0 \subset X$ and a mapping $S : Y_0 \to Y_0$ such that

1. $\forall x \in X_0,\ T^n x \to 0;$
2. $\forall y \in Y_0,\ S^n y \to 0;$
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S. Grivaux (2005): NO!

Classes of hypercyclic operators

The first hypercyclic operator on a Banach space is due to Rolewicz:

$$T : X \to X, (x_n)_n \to (x_{n+1})_n$$

where

$$X = \ell^p_{(|\lambda|^{-n})_n} = \Big\{ (x_n)_{n\geq 0} : \sum_{n=0}^{\infty} |x_n|^p \frac{1}{|\lambda|^n} < \infty \Big\}, \quad |\lambda| > 1, 1 \leq p < \infty.$$

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Since then, many composition operators have turned out to be hypercyclic:

- backward shift operators on sequence spaces
- composition operators on spaces of analytic functions
- composition operators on spaces of continuous functions

And for many of these classes, hypercyclicity has been characterized.

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$$\mu(A \setminus B_k) \to 0, \quad \mu(f^{-n_k}(B_k)) \to 0 \quad \textit{and} \quad \mu(f^{n_k}(B_k)) \to 0.$$

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T_f satisfies Kitai's criterion if and only if, for every measurable set A of finite measure and every $\varepsilon > 0$ there is a measurable subset $B \subset A$ such that

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Answer: NO!

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Theorem (Gomes, G-E)

There is a σ-finite measure space $(X, \mathcal{B}, \mu)$ and a composition operator

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that is mixing but does not satisfy Kitai's criterion ($1 \leq p < \infty$).

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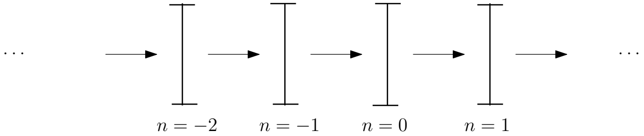
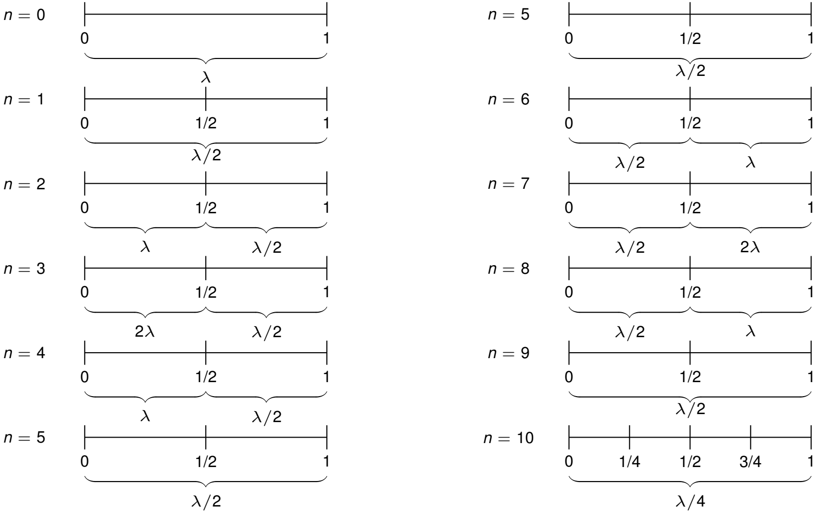
The space is

$$X = \bigcup_{n=-\infty}^{\infty} \Big(\{n\} \times [0, 1] \Big)$$

and the map is

$$f : (n, x) \to (n + 1, x).$$




n = 0
0
1
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n = 1
0
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n = 3
0
1/2
1
2λ
λ/2
n = 4
0
1/2
1
λ
λ/2
n = 5
0
1/2
1
λ/2
n = 5
0
1/2
1
λ/2
n = 6
0
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n = 7
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n = 8
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1
λ/2
λ
n = 9
0
1/2
1
λ/2
n = 10
0
1/4
1/2
3/4
1
λ/4

T_f is mixing: $\forall A$, $\mu(A) < \infty$, $\exists B_n \subset A$

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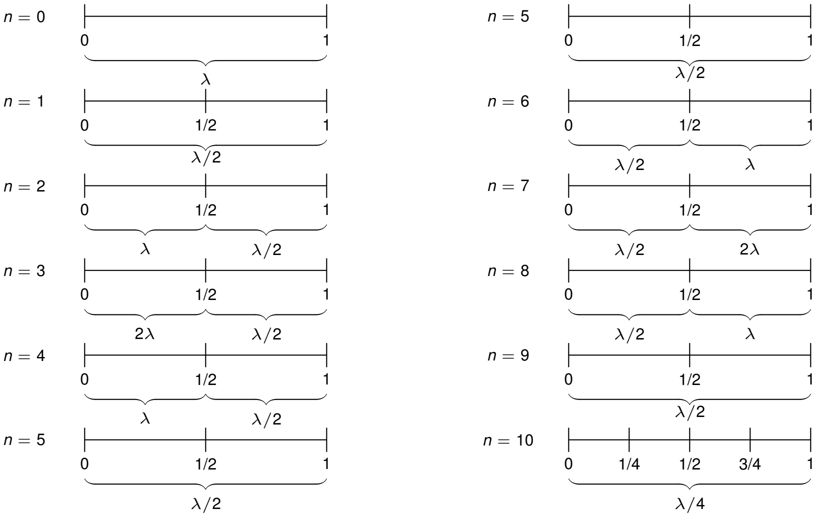

T_f does not satisfy Kitai: $\forall A$, $\mu(A) < \infty$, $\forall \varepsilon > 0$ $\exists B \subset A$

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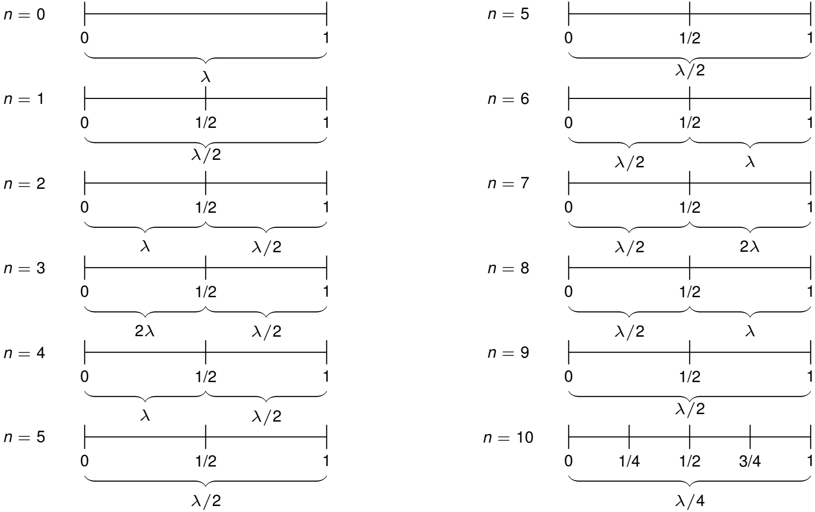

Final remarks:

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• Our result seems to be the first characterization of when Kitai holds (where Kitai $\neq$ mixing).

• Darji, Pires (2021) characterize when the Frequent Hypercyclicity Criterion = Chaoticity Criterion holds ($p \geq 2$).

F. Bayart, U. B. Darji, and B. Pires, Topological transitivity and mixing of composition operators, *J. Math. Anal. Appl.* 465 (2018), 125–139.

U. B. Darji and B. Pires, Chaos and frequent hypercyclicity for composition operators, *Proc. Edinb. Math. Soc.*, II. Ser. 64 (2021), 513–531.

D. Gomes and K.-G. Grosse-Erdmann, Kitai's criterion for composition operators, in preparation.

S. Grivaux, Hypercyclic operators, mixing operators and the Bounded Steps Problem. *Journal of Operator Theory*, 54 (2005), 147–168.

C. Kitai, *Invariant closed sets for linear operators*, Ph.D. thesis, University of Toronto, Toronto, 1982.