Chaos on Peano continua

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Joint work with Benjamin Vejnar, work in progress

TDS

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Regarding chaos, there are three most widely accepted notions of chaos: chaos according to Devaney, chaos according to Li-Yorke and (positive) topological entropy.

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- Fact: f is transitive iff it has a transitive point iff the set of all transitive points is dense G_δ in X.
- We say that f is LEO (locally eventually onto) if for every nonempty, open $U \subseteq X$ there exists n natural s. t. $f^n(U) = X$.

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- We say that f has a sensitive dependence on initial conditions if there exists $\delta > 0$ such that for every $x \in X$ there exists $y \in X, d(x, y) < \delta$ and n natural such that $d(f^n(x), f^n(y)) \geq \delta$.

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- We say that a pair of points $x, y \in X$ is scrambled if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

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- We say that a set $S \subseteq X$ is scrambled if for every $x \neq y \in S$ the pair x, y is scrambled.

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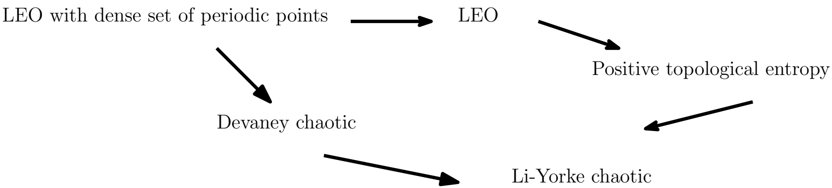
- Positive topological entropy

Chaos on compact spaces



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- A metric space is a Peano continuum iff it is a continuous image of $I = [0, 1]$,
- A continuum X is a Peano continuum iff it has the property S, i.e. iff for every $\varepsilon > 0$ there exists a finite cover of X formed by connected sets with diameter smaller than ε.

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The work is in progress.

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Proof sketch:
Let X be a Peano continuum. We will find maps $g : X \to [0,1]$ and $f : [0,1] \to X$ such that $f \circ g : X \to X$ is LEO with dense set of periodic points.
Firstly we construct the map g, this is the easy part as we do not require much; we only need that g sends no nonempty open set to a point.
Secondly we construct f. By an easy modification of the proof of the Hahn–Mazurkiewicz Theorem it is possible to ensure that $f \circ g$ is LEO. The dense set of periodic points requires some more control.

The End

Thank you for your attention!

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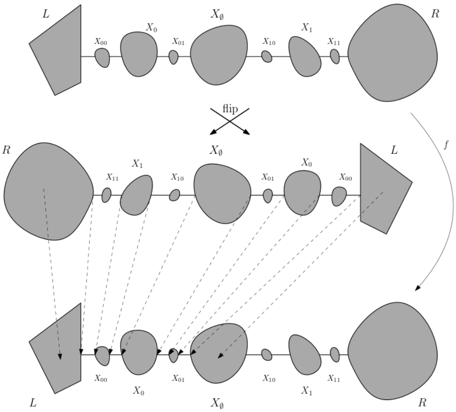

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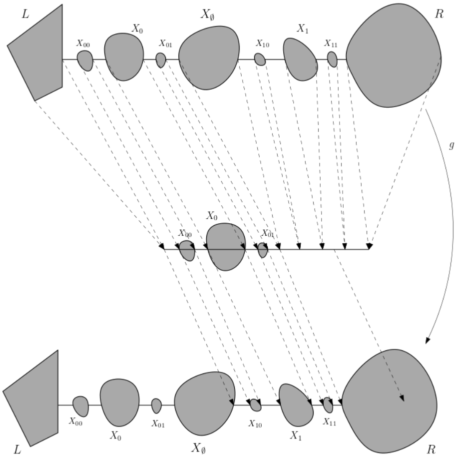

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