Compactness and averaging operators on function spaces

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Outline

Introduction

Throughout this talk, $X = (X, \mathcal{M}, \mu)$ is a metric measure space satisfying the following:

- (Borel) $\mathcal{M}$ contains the Borel sets of X;
- (Borel-regular) Each subset $E \subset X$ is contained in some Borel set $B \subset X$ s.t. $\mu(B \setminus E) = 0$;
- For ${}^{\forall}x \in X$ and ${}^{\forall}r \in (0, \infty)$, $0 < \mu(B(x, r)) < \infty$.

For $p \in [1, \infty)$, let $\mathrm{L}^{\mathrm{p}}(X) = (\mathrm{L}^{\mathrm{p}}(X), \|\cdot\|_p)$ be the Lebesgue space on X.

$\mathcal{M}$ is the collection of measurable sets in X and μ is a measure on X.
$B(x, r) \subset X$ is the closed ball centered at x with radius r.

A normed linear space $E(X) = (E(X), \|\cdot\|_E) \subset L^0(X)$ is a *Banach function space* on X if for ${}^\forall f, {}^\forall g \in E(X), {}^\forall \{f_n\} \subset E(X)$,

(B1) $\|f\|_E = \||f|\|_E$, and $\|f\|_E = 0 \Leftrightarrow f = 0$ a.e.;

(B2) $0 \le g \le f$ a.e. $\Rightarrow \|g\|_E \le \|f\|_E$;

(B3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_E \uparrow \|f\|_E$;

and for ${}^\forall A \in \mathcal{M}$ with $\mu(A) < \infty$,

(B4) $\chi_A \in E(X)$ and $\|\chi_A\|_E < \infty$;

(B5) ${}^\exists \alpha(A) \in (0, \infty)$ s.t. $\int_A |f| d\mu \le \alpha(A)\|f\|_E$ for ${}^\forall f \in E(X)$.

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χ_A is the characteristic function of $A \subset X$.

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Example (Lebesgue space)
$L^p(X)$ is a Banach function space.

(B5) By Hölder's inequality, for ${}^\forall A \in \mathcal{M}$,

$$\int_A |f(x)| d\mu(x) \le (\mu(A))^{1-1/p} \|f\|_p,$$

so we can choose $\alpha(A) = (\mu(A))^{1-1/p}$.

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Theorem 1 (A.N. Kolmogorov (1931), M. Riesz (1933))

$F \underset{bounded}{\subset} \mathrm{L}^{\mathrm{p}}(\mathbb{R}^n)$ *is relatively compact* $\Leftrightarrow$

1. *For* ${}^\forall\epsilon > 0$, ${}^\exists\delta > 0$ *s.t.* $\|\tau_a f - f\|_p < \epsilon$ *for* ${}^\forall f \in F$ *and* ${}^\forall a \in \mathbb{R}^n$ *with* $|a| < \delta$.
2. *For* ${}^\forall\epsilon > 0$, ${}^\exists r > 0$ *s.t.* $\|f\chi_{\mathbb{R}^n \setminus B(\mathbf{0},r)}\|_p < \epsilon$ *for* ${}^\forall f \in F$.

For $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, $\tau_a f(x) = f(x - a)$.

Fix $r > 0$ and define $A_r : E(X) \to L^0(X)$ by

$$A_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y),$$

which is called the *averaging operator* on $E(X)$.

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Problem

Give a criterion for subsets of $\mathrm{L}^{\mathrm{p}}(X)$ to be compact.

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An operator $T : E_1 \to E_2$ between normed linear spaces is *compact* if for $^\forall A \underset{bounded}{\subset} E_1$, $T(A)$ is relatively compact in E_2.

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Problem

Give a sufficient and necessary condition for $A_r : E(X) \to E(X)$ to be compact.

For $^\forall f \in L^0(X)$, $A_r f \in L^0(X)$.

Compact sets in Lebesgue spaces

X is doubling if ${}^{\exists}\gamma \geq 1$ s.t. for ${}^{\forall}x \in X$ and ${}^{\forall}r > 0$,

$$\mu(B(x, 2r)) \leq \gamma\mu(B(x, r)).$$

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Theorem 2 (P. Górka-A. Macios (2014))

Let X be doubling and $p \in (1,\infty)$. Suppose that

$(\bullet)$ *for ${}^{\forall}r > 0$,*

$$\inf\{\mu(B(x,r)) \mid x \in X\} > 0.$$

Then $F \underset{bounded}{\subset} \mathrm{L}^{\mathrm{p}}(X)$ is relatively compact $\Leftrightarrow$

1. *For ${}^{\forall}\epsilon > 0$, ${}^{\exists}\delta > 0$ s.t. for ${}^{\forall}f \in F$ and ${}^{\forall}r \in (0,\delta)$, $\|A_r f - f\|_p < \epsilon$.*
2. *For ${}^{\forall}\epsilon > 0$, ${}^{\exists}E \underset{bounded}{\subset} X$ s.t. $\|f\chi_{X\setminus E}\|_p < \epsilon$ for ${}^{\forall}f \in F$.*

For $r > 0$, X is *r-doubling* if $^{\exists}\gamma \geq 1$ s.t. for $^{\forall}x \in X$,

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Remark (doubling)
X is doubling $\Rightarrow$ X is r-doubling for ${}^{\forall}r > 0$.

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X has *the Vitali covering property (abbrev. VCP)* if

- Let ${}^{\forall}A \subset X$ and $\mathcal{B}$ be any family of closed balls centered in A with uniformly bounded radii s.t. for ${}^{\forall}x \in A$, ${}^{\exists}r > 0$ s.t. $B(x, r) \in \mathcal{B}$ and $\inf\{s > 0 \mid B(x, s) \in \mathcal{B}\} = 0$. Then ${}^{\exists}\mathcal{B}' \subset \mathcal{B}$ consisting of pairwise disjoint countable closed balls s.t.

$$\mu(A \setminus \bigcup \mathcal{B}') = 0.$$

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Theorem (Vitali)
X is doubling $\Rightarrow$ X has the VCP.

Theorem A (K. (2023))

Let X be r-doubling for ${}^\forall r > 0$ and having the VCP. Suppose that

(∗) for ${}^\forall x \in X$ and ${}^\forall r > 0$,

$$\mu(B(x,r) \triangle B(y,r)) \to 0 \text{ as } y \to x.$$

Then $F \underset{bounded}{\subset} \mathrm{L^p}(X)$ is relatively compact $\Leftrightarrow$

1. For ${}^\forall \epsilon > 0$, ${}^\exists \delta > 0$ s.t. for ${}^\forall f \in F$ and ${}^\forall r \in (0, \delta)$, $\|A_r f - f\|_p < \epsilon$.
2. For ${}^\forall \epsilon > 0$, ${}^\exists E \underset{bounded}{\subset} X$ s.t. $\|f \chi_{X \setminus E}\|_p < \epsilon$ for ${}^\forall f \in F$.

For $A, B \subset X$, let $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

($\star$) For $^\forall x \in X$,

$$(0,\infty) \ni r \mapsto \mu(B(x,r)) \in (0,\infty)$$

is continuous.

($*$) For $^\forall x \in X$ and $^\forall r \in (0,\infty)$,

$$\mu(B(x,r) \triangle B(y,r)) \to 0 \text{ as } y \to x.$$

($\dagger$) For $^\forall r \in (0,\infty)$,

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Then ($\star$) $\Rightarrow$ ($*$) $\Rightarrow$ ($\dagger$).

($\bullet$) For ${}^\forall r > 0$,

$$\inf\{\mu(B(x,r)) \mid x \in X\} > 0.$$

There are no implications between ($\bullet$) and ($*$).

Remark (Hilbert space)
X is infinite and separable $\Rightarrow L^p(X) \approx \ell_2$.

ℓ_2 is the separable Hilbert space, ℓ_2^f is the linear span of the canonical orthonormal basis on ℓ_2, and $\mathbf{Q}$ is the Hilbert cube.

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Let

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Let

$$\mathrm{LIP}_b(X) = \{f \in \mathrm{L^p}(X) \mid f \text{ is Lipschitz with a bounded support}\}.$$

Corollary (K. (2023))

Let X be non-degenerate, separable, r-doubling for ${}^\forall r > 0$ and having the VCP. Suppose that

($\star$) for ${}^\forall x \in X$,

$$(0, \infty) \ni r \mapsto \mu(B(x, r)) \in (0, \infty)$$

is continuous.

Then $(\mathrm{L^p}(X), \mathrm{LIP}_b(X)) \approx (\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$.

ℓ_2 is the separable Hilbert space, ℓ_2^f is the linear span of the canonical orthonormal basis on ℓ_2, and $\mathbf{Q}$ is the Hilbert cube.

Compactness of averaging operators

$\| \cdot \|_E$ is *absolutely continuous with respect to* 1 if $\|\chi_{A_i}\|_E \to 0$ for ${}^\forall\{A_i\} \subset \mathcal{M}$ with $\chi_{A_i} \to 0$ a.e..

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Theorem B (K. (202?))

Let $r > 0$. Suppose that $A_r(E(X)) \subset E(X)$ and

1. X has the VCP;
2. $\inf\{\mu(B(x,r)) \mid x \in X\} > 0$;
3. for $^\forall x \in X$, $\mu(B(y,r)) \to \mu(B(x,r))$ and $\alpha(B(x,r) \triangle B(y,r)) \to 0$ as $y \to x$;
4. $\|\cdot\|_E$ is absolutely continuous with respect to 1.

Then $\mu(X) < \infty \Rightarrow A_r$ is compact.

Theorem C (K. (202?))

Let $r > 0$. Suppose that $A_r(E(X)) \subset E(X)$ and

1. X is r-doubling;
2. $\sup\left\{ \left\| \frac{\alpha(B(x,r))\chi_{B(x,2r)}}{\mu(B(x,r))} \right\|_E \,\middle|\, x \in X \right\} < \infty$ for ${}^{\exists}\alpha$ as in (B5).

Then A_r is compact $\Rightarrow$ X is bounded.

Corollary (K. (202?))

Let $p, q \in (1, \infty)$ and $r > 0$. Suppose that

1. X has the VCP;
2. X is s-doubling for ${}^\forall s \geq r$;
3. for ${}^\forall x \in X$, $\mu(B(x, r) \triangle B(y, r)) \to 0$ as $y \to x$.

Then $A_r : L^{p,q}(X) \to L^{p,q}(X)$ is compact $\Leftrightarrow$ X is bounded.

$L^{p,q}(X)$ is the Lorentz space on X.