On property (B) in function spaces

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SUMTOPO 2024

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- $C_k(X) \equiv$ compact–open topology
- $C_w(K) \equiv$ weak topology on a Banach space $C(K)$ for compact K

Motivation

General problem

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We will say that a topological space X has property (B) if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of closed nowhere dense subsets of X absorbing all compact subsets of X i.e. for each compact $K \subset X$ there is $n \in \mathbb{N}$ s.t. $K \subset A_n$.

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GERLITS & PYTKEEV

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Problem (Krupski–Marciszewski, 2017)

Characterize all Tychonoff spaces X s.t. the space of real–valued continuous functions with the pointwise convergence topology $C_p(X)$ satisfies property (B).

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A family $\mathcal{A}$ of subsets of a space X is **point–finite**, if each point $x \in X$ lies only in finitely many elements of $\mathcal{A}$.

Another piece of context

A topological space Z is

- a Fréchet–Urysohn space iff for any $A \subset Z$ and $x \in \mathrm{cl}(A)$ there is a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converging to x.
- a κ–Fréchet–Urysohn space iff for any **open** $U \subset Z$ and $x \in \mathrm{cl}(U)$ there is a sequence $(x_n)_{n \in \mathbb{N}} \subset U$ converging to x.

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Main result for C_p–spaces

Proposition

Let Z be any space. If Z is κ–Fréchet–Urysohn then Z does not have the property (B).

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Moreover if X is compact — or even more general: is Čech complete — then all conditions above are equivalent to X being scattered.

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Theorem (Gruenhage–Ma)

If X is locally compact then TFAE:

1. Every moving off family of nonempty compact subsets of X has a countable infinite subfamily with a discrete open expansion
2. $C_k(X)$ is Baire

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Question 1 (Tkachuk)

Suppose that all functionally bounded subsets of X are finite. Must X have the property (κ)?

Quick digression on the weakening of the density topology

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The space $(\mathbb{R}, \tau)$ is Tychonoff, it doesn't satisfy property (κ) and all τ–functionally bounded sets are finite.

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Yes. If $C_p(X)$ does not satisfy (B) then X has (κ). By another theorem of Tkachuk a first countable pseudocompact space with property (κ) has to be scattered.