# Function spaces on Corson-like compacta

Krzysztof Zakrzewski

University of Warsaw

July 8, 2024

All topological spaces are Tikhonov.

All topological spaces are Tikhonov.

## Definition

A space $K$ is an Eberlein compact space if $K$ is homeomorphic to a weakly compact subset of a Banach space.

All topological spaces are Tikhonov.

## Definition

A space $K$ is an Eberlein compact space if $K$ is homeomorphic to a weakly compact subset of a Banach space.

Equivalently, a compact space $K$ is an Eberlein compactum if $K$ can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

for some set $\Gamma$.

All topological spaces are Tikhonov.

## Definition

A space $K$ is an Eberlein compact space if $K$ is homeomorphic to a weakly compact subset of a Banach space.

Equivalently, a compact space $K$ is an Eberlein compactum if $K$ can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

for some set $\Gamma$.
All metrizable compacta are Eberlein compact spaces.

All topological spaces are Tikhonov.

## Definition

A space $K$ is an Eberlein compact space if $K$ is homeomorphic to a weakly compact subset of a Banach space.

Equivalently, a compact space $K$ is an Eberlein compactum if $K$ can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

for some set $\Gamma$.

All metrizable compacta are Eberlein compact spaces.

Continuous images, closed subspaces, countable products of Eberlein compacta are Eberlein compact spaces.

A compact space $K$ is Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma$-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

A compact space $K$ is Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma$-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

Clearly, the class of Corson compact spaces contains all Eberlein compacta.

A compact space $K$ is Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma$-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

Clearly, the class of Corson compact spaces contains all Eberlein compacta.
Let $\kappa$ be an infinite cardinal number. A compact space $K$ is $\kappa$-Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma_\kappa$-product of real lines

$$\Sigma_\kappa(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| < \kappa\}.$$

A compact space $K$ is Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma$-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

Clearly, the class of Corson compact spaces contains all Eberlein compacta.
Let $\kappa$ be an infinite cardinal number. A compact space $K$ is $\kappa$-Corson compact if, for some set $\Gamma$, $K$ is homeomorphic to a subset of the $\Sigma_\kappa$-product of real lines

$$\Sigma_\kappa(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| < \kappa\}.$$

Obviously, the class of Corson compact spaces coincides with the class of $\omega_1$-Corson compact spaces.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let $a_\gamma$ be a fixed point of $X_\gamma$.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let $a_\gamma$ be a fixed point of $X_\gamma$.
The $\sigma$-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is the following subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}\,.$$

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let $a_\gamma$ be a fixed point of $X_\gamma$.
The $\sigma$-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is the following subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}.$$

If $X_\gamma = I = [0, 1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the $\sigma$-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let $a_\gamma$ be a fixed point of $X_\gamma$.
The $\sigma$-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is the following subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}\,.$$

If $X_\gamma = I = [0,1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the $\sigma$-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.
If $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then we denote the $\sigma$-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I^\omega, \Gamma)$.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let $a_\gamma$ be a fixed point of $X_\gamma$.

The $\sigma$-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is the following subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}.$$

If $X_\gamma = I = [0,1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the $\sigma$-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.

If $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then we denote the $\sigma$-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I^\omega, \Gamma)$.

For $\kappa = \omega$, $\Sigma_\kappa(\mathbb{R}^\Gamma) = \sigma(\mathbb{R}, \Gamma)$.

A compact space $K$ is NY compact if $K$ can be embedded into some $\sigma$-product of metrizable compacta.

A compact space $K$ is NY compact if $K$ can be embedded into some $\sigma$-product of metrizable compacta.
We denote the class of NY compact spaces by $\mathcal{NY}$.

A compact space $K$ is NY compact if $K$ can be embedded into some $\sigma$-product of metrizable compacta.
We denote the class of NY compact spaces by $\mathcal{NY}$.
Every $NY$ compact space is Eberlein compact.

A compact space $K$ is NY compact if $K$ can be embedded into some $\sigma$-product of metrizable compacta.
We denote the class of NY compact spaces by $\mathcal{NY}$.
Every $NY$ compact space is Eberlein compact.

## Proposition

*For a compact space $K$ we have*

- (a) *$K$ is $\omega$-Corson if and only if it can be embedded into some $\sigma$-product of metrizable finitely dimensional compacta if and only if it can be embedded into the $\sigma$-product $\sigma(I, \Gamma)$ for some set $\Gamma$.*
- (b) *$K$ is NY compact if and only if it can be embedded into the $\sigma$-product $\sigma(I^\omega, \Gamma)$ for some set $\Gamma$.*

A space $X$ is metacompact if every open cover of $X$ has a point-finite open refinement.

A space $X$ is metacompact if every open cover of $X$ has a point-finite open refinement.

**Theorem (Marciszewski, Plebanek, Z.)**

*For a compact space $K$, the following conditions are equivalent:*

- (a) *$K$ is $\omega$-Corson;*
- (b) *$K$ is hereditarily metacompact and each nonempty subspace $A$ of $K$ contains a nonempty relatively open separable, metrizable, finitely dimensional subspace $U$.*

### Theorem ( Marciszewski, Plebanek, Z.)

*For a compact space $K$, the following conditions are equivalent:*

(a) *$K$ belongs to the class $\mathcal{NY}$;*

(b) *$K$ is hereditarily metacompact and each nonempty subspace $A$ of $K$ contains a nonempty relatively open subspace $U$ of countable weight.*

## Theorem ( Marciszewski, Plebanek, Z.)

*For a compact space $K$, the following conditions are equivalent:*

(a) *$K$ belongs to the class $\mathcal{NY}$;*

(b) *$K$ is hereditarily metacompact and each nonempty subspace $A$ of $K$ contains a nonempty relatively open subspace $U$ of countable weight.*

## Proposition (Nakhmanson and Yakovlev)

*The class $\mathcal{NY}$ is stable under continuous images*

## Theorem ( Marciszewski, Plebanek, Z.)

*For a compact space $K$, the following conditions are equivalent:*

(a) *$K$ belongs to the class $\mathcal{N}\mathcal{Y}$;*

(b) *$K$ is hereditarily metacompact and each nonempty subspace $A$ of $K$ contains a nonempty relatively open subspace $U$ of countable weight.*

## Proposition (Nakhmanson and Yakovlev)

*The class $\mathcal{N}\mathcal{Y}$ is stable under continuous images*

For a space $X$, $C_p(X)$ denotes the space of real continuous functions on $X$ endowed with the pointwise convergence topology.

## Theorem ( Marciszewski, Plebanek, Z.)

*For a compact space $K$, the following conditions are equivalent:*

(a) *$K$ belongs to the class $\mathcal{NY}$;*

(b) *$K$ is hereditarily metacompact and each nonempty subspace $A$ of $K$ contains a nonempty relatively open subspace $U$ of countable weight.*

## Proposition (Nakhmanson and Yakovlev)

*The class $\mathcal{NY}$ is stable under continuous images*

For a space $X$, $C_p(X)$ denotes the space of real continuous functions on $X$ endowed with the pointwise convergence topology.

## Theorem (Z.)

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $NY$ compact, then $L$ is $NY$ compact as well.*

### Definition

For a continuous linear operator $T : X \to Y$ between two linear topological spaces, the dual operator $T^* : X^* \to Y^*$ is given by the formula $T(\phi) = \phi \circ T$.

## Definition

For a continuous linear operator $T : X \to Y$ between two linear topological spaces, the dual operator $T^* : X^* \to Y^*$ is given by the formula $T(\phi) = \phi \circ T$.

## Lemma

*Let $T : X \to Y$ be a continuous linear operator between two locally convex linear topological spaces, then*
*$T(X)$ is dense in $Y \iff T^*$ is an injection.*

### Definition

For a continuous linear operator $T : X \to Y$ between two linear topological spaces, the dual operator $T^* : X^* \to Y^*$ is given by the formula $T(\phi) = \phi \circ T$.

### Lemma

*Let $T : X \to Y$ be a continuous linear operator between two locally convex linear topological spaces, then*
*$T(X)$ is dense in $Y \iff T^*$ is an injection.*

### Lemma

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $NY$ compact, then $L$ is a union of countably many $NY$ compact spaces.*

## Lemma

*Let $K$ be a compact space such that $K = \bigcup_{n\in\mathbb{N}} K_n$ where $\{K_n : n \in \mathbb{N}\}$ is a sequence of NY compact spaces. Then every subspace of $K$ contains a relatively open subspace of countable weight.*

## Lemma

*Let $K$ be a compact space such that $K = \bigcup_{n\in\mathbb{N}} K_n$ where $\{K_n : n \in \mathbb{N}\}$ is a sequence of $NY$ compact spaces. Then every subspace of $K$ contains a relatively open subspace of countable weight.*

## Proof.

Assume there exists $A \subseteq K$ such that every relatively open $U \subseteq A$ has uncountable weight, then $\overline{A}$ has the same property. Indeed, assume there is a nonempty, relatively open $U \subseteq \overline{A}$ of countable weight, then $U \cap A \neq \emptyset$, and therefore $U \cap A \subseteq A$ is a relatively open, nonempty subset of $A$ with $w(U \cap A) \leq \omega$, contradiction.
Without loss of generality, we can assume that $A$ is closed and therefore compact. Then it has the same property as $K$, so we can assume that $A = K$. As $K = \bigcup_{n\in\mathbb{N}} K_n$, by the Baire category theorem, there is $K_n$ with nonempty interior. Since $K_n$ is $NY$ compact, there exists a nonempty, open $V \subseteq int_K(K_n)$ with $w(V) \leq \omega$. Then $V$ is an open subset of $K$ of countable weight. $\square$

## Theorem

*For a compact space $K$, the following conditions are equivalent:*
*a) $K$ is Eberlein compact;*
*b) $C_p(K)$ has a $\sigma$-compact dense subspace.*

## Theorem

*For a compact space $K$, the following conditions are equivalent:*
*a) $K$ is Eberlein compact;*
*b) $C_p(K)$ has a $\sigma$-compact dense subspace.*

## Definition

A topological space is called $\sigma$-metacompact iff every open cover has an open refinement which is a countable union of point finite families.

### Theorem

*For a compact space $K$, the following conditions are equivalent:*
*a) $K$ is Eberlein compact;*
*b) $C_p(K)$ has a $\sigma$-compact dense subspace.*

### Definition

A topological space is called $\sigma$-metacompact iff every open cover has an open refinement which is a countable union of point finite families.

### Theorem (Gruenhage)

*For a compact space $K$, the following conditions are equivalent:*
*a) $K$ is Eberlein compact;*
*b) $K^2$ is hereditarily $\sigma$-metacompact.*

## Lemma

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $NY$ compact, then $L$ is hereditarily $\sigma$ - metacompact.*

## Lemma

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is NY compact, then $L$ is hereditarily $\sigma$ - metacompact.*

## Proof.

Space $K$ is Eberlein compact, so there exists a dense $\sigma$-compact set $D \subset C_p(K)$. Then $T(D)$ is again a dense $\sigma$-compact subset of $C_p(L)$. Consequently, space $L$ is Eberlein compact and therefore hereditarily $\sigma$-metacompact. □

## Lemma

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $NY$ compact, then $L$ is hereditarily $\sigma$ - metacompact.*

## Proof.

Space $K$ is Eberlein compact, so there exists a dense $\sigma$-compact set $D \subset C_p(K)$. Then $T(D)$ is again a dense $\sigma$-compact subset of $C_p(L)$. Consequently, space $L$ is Eberlein compact and therefore hereditarily $\sigma$-metacompact. $\square$

Concluding, space $L$ is hereditarily $\sigma$ - metacompact and is a union of countably many $NY$ compact spaces.

## Lemma

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $NY$ compact, then $L$ is hereditarily $\sigma$ - metacompact.*

## Proof.

Space $K$ is Eberlein compact, so there exists a dense $\sigma$-compact set $D \subset C_p(K)$. Then $T(D)$ is again a dense $\sigma$-compact subset of $C_p(L)$. Consequently, space $L$ is Eberlein compact and therefore hereditarily $\sigma$-metacompact. □

Concluding, space $L$ is hereditarily $\sigma$ - metacompact and is a union of countably many $NY$ compact spaces.

## Lemma

*A $\sigma$-metacompact space which is a union of countably many closed, metacompact subspaces is metacompact.*

**Definition**

A topological space is strongly countably dimensional iff it is a countable union of finitely dimensional closed subspaces.

### Definition

A topological space is strongly countably dimensional iff it is a countable union of finitely dimensional closed subspaces.

### Theorem ( Marciszewski, Plebanek, Z.)

*An NY compact space $K$ is $\omega$-Corson compact if and only if it is strongly countably dimensional.*

**Theorem (Z.)**

*Let $X$ and $Y$ be $\sigma$-compact spaces. Assume there exists a continuous linear transformation $T : C_p(X) \to C_p(Y)$ such that $T(C_p(X))$ is dense in $C_p(Y)$. If $X$ is strongly countably dimensional, then $Y$ is strongly countably dimensional as well.*

**Theorem (Z.)**

*Let $X$ and $Y$ be $\sigma$-compact spaces. Assume there exists a continuous linear transformation $T : C_p(X) \to C_p(Y)$ such that $T(C_p(X))$ is dense in $C_p(Y)$. If $X$ is strongly countably dimensional, then $Y$ is strongly countably dimensional as well.*

**Theorem (Z.)**

*Let $K$ and $L$ be compact spaces. Assume there exists a continuous linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If $K$ is $\omega$ - Corson compact, then $L$ is $\omega$ - Corson compact as well.*