Topological remarks on end and edge-end spaces

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This is a joint work with Paulo Magalhães Júnior and Lucas Real.

A bit of motivation

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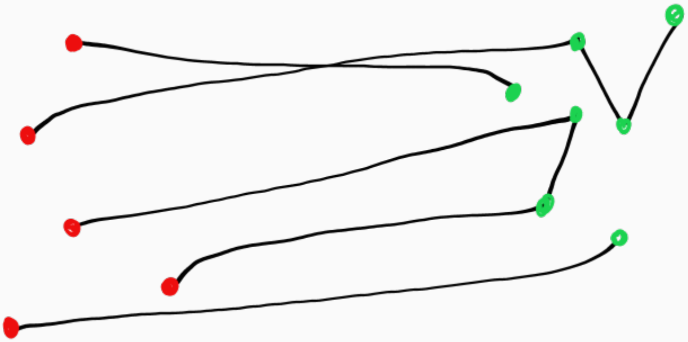
Separation



Vertex-Menger

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Theorem (Menger)
Given $A, B \subset V(G)$, the smallest size of a set $F \subset V(G)$ such that A, B are not connected in $G \setminus F$ is the largest number of disjoint paths connecting A and B.

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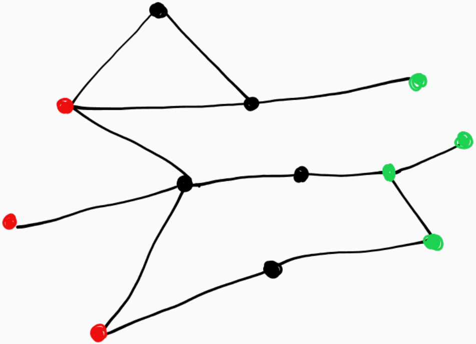

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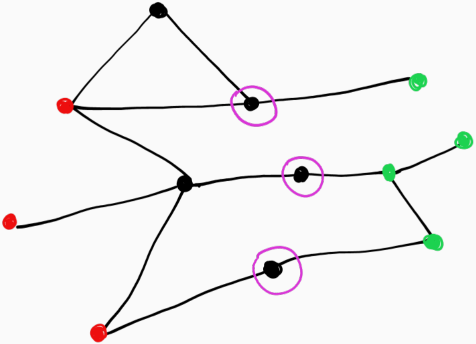

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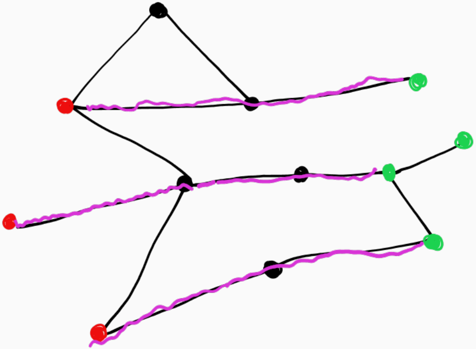

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Theorem (Menger)
Given $A, B \subset V(G)$, the smallest size of a set $F \subset E(G)$ such that A, B are not connected in $G \setminus F$ is the largest number of edge-disjoint paths connecting A and B.

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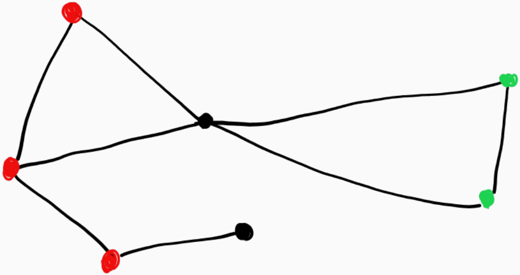

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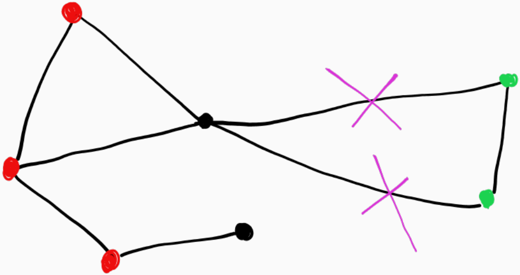

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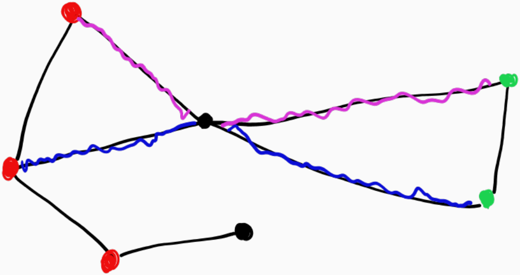

Infinite graphs

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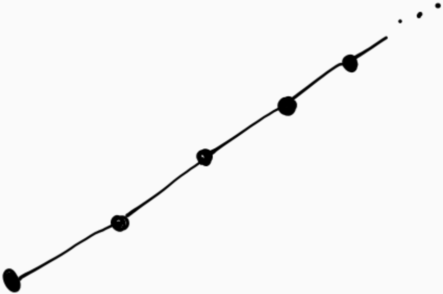
For infinite graphs, an important concept is a ray, which is an infinite sequence of distinct vertices, each joined to the next one by an edge.

A ray



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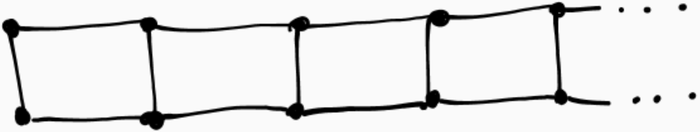
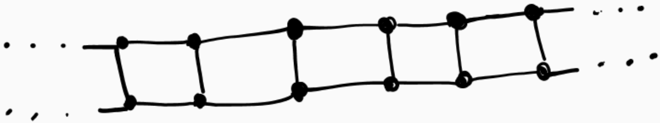
An end is each equivalence class of this relation.

One end



Two ends



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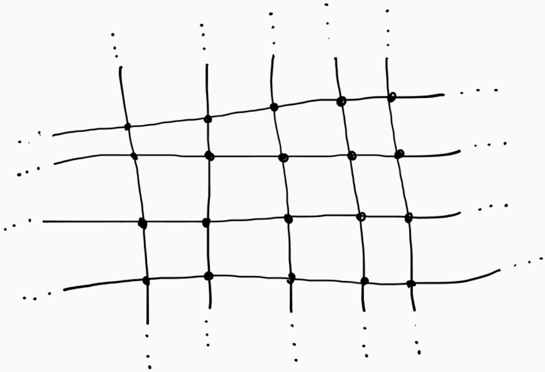

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Notice that there is a "topological hypothesis" here.

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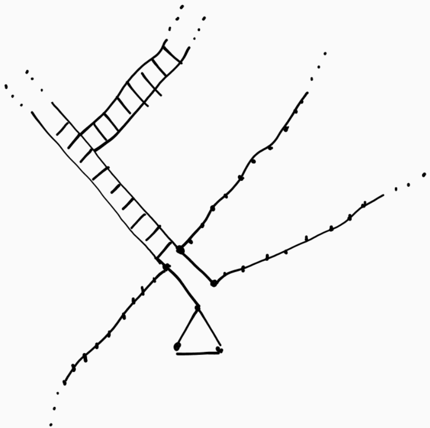
Note that the particular r taken is not important. The neighborhood is determined by the finite set that is removed.

A basic open neighborhood



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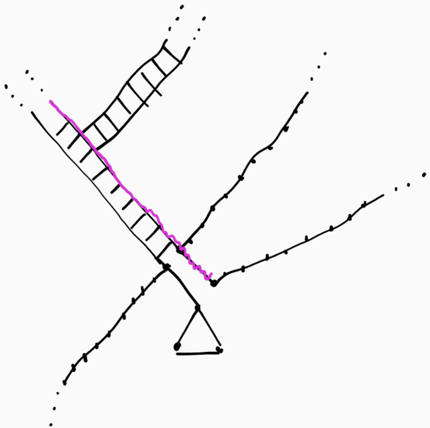

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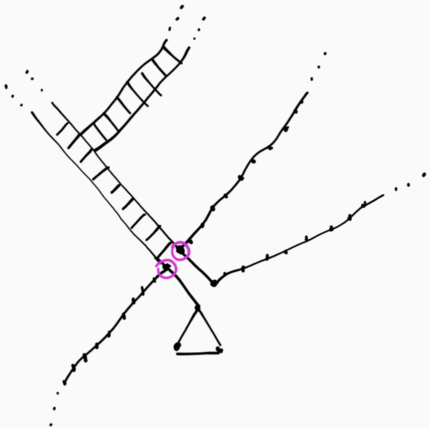

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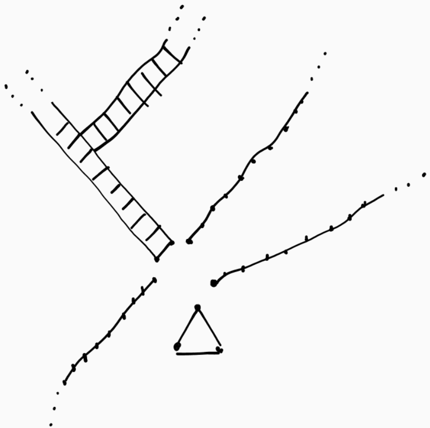

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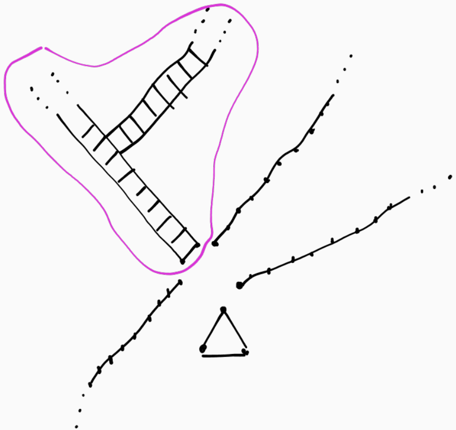

The end space

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Equipped with this topology, $\Omega(G)$ is called the end space associated to G (it is important to notice that G is not a subset of $\Omega(G)$ - sometimes another space is considered, whose points are $G \cup \Omega(G)$, but we are not working with them today).

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- Regular;
- Ultraparacompact;
- Metrizable iff G has a normal end-faithful tree.

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After, Pitz used this representation to answer Diestel's question in [4].

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A family is complete if every centered subfamily has non-empty intersection. A family $\mathcal{C}$ is hereditarily complete if, for every closed set F, $\{C \cap F : C \in \mathcal{C}\}$ is complete.

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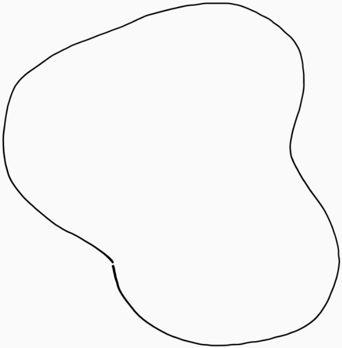

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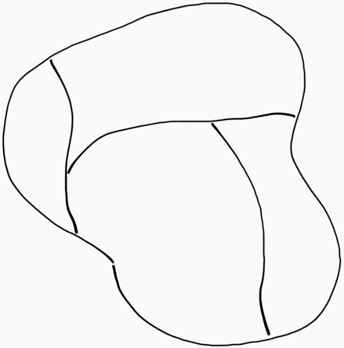

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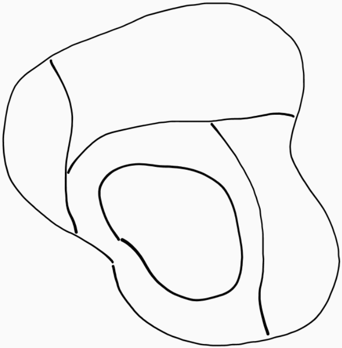

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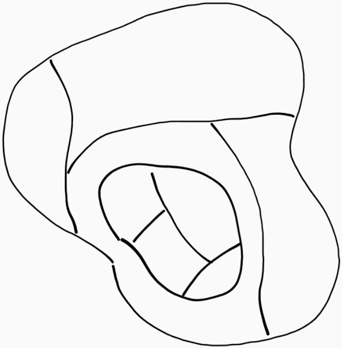

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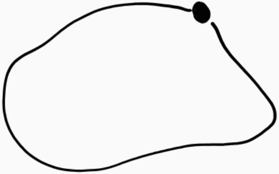

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- *$\mathcal{C}$ is noetherian, i.e., every $\subset$-increasing family of elements of $\mathcal{C}$ has a maximum;*
- *Player II has a winning strategy in the $End_{\mathcal{C}}$ game.*

What do we get from it?

One advantage of using the game in this characterization is that from the winning strategy it is simple to get a “nice” tree as in the representation result of Kurkofka and Pitz.

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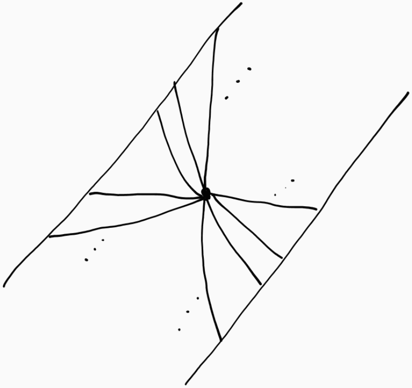
If we drop the locally finite assumption, everything changes.

An example



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It should be pointed out that, for the locally finite case, these two definitions are equivalent.

Who are these topological spaces?

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Then again we have the question about what are the topological spaces obtained in this form.

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This inclusion is proper (we will see this in a bit).

Representation

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Proposition (Aurichi, Magalhães Júnior, Real)
In the point of view of topological spaces,

$$\{\Omega_E(G) : G \text{ is a graph}\} = \{\Omega(G) : G \text{ is a graph such that}$$

$$\text{any vertex edge-dominates at most one end}\}.$$

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This allows us to show that not every end space is an edge-end space.

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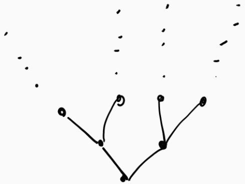
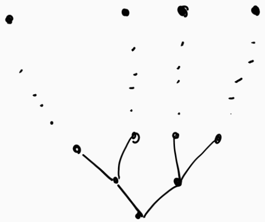
Proof.
It is a compact (therefore Lindelöf), first countable space. But it is not metrizable. □

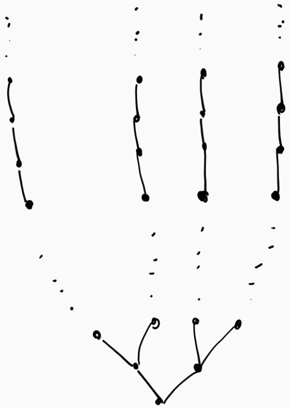
Final example



Final example



Final example



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Thank you for your attention!