Renormalization in Lorenz maps – completely invariant sets and periodic orbits

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The talk is based on a joint work with Piotr Oprocha

Ł. Cholewa, P. Oprocha, *Renormalization in Lorenz maps – completely invariant sets and periodic orbits*, preprint, arXiv:2104.00110.

Presentation plan

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- Introduction

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- Theorem of Yiming Ding: Renormalizations and invariant sets

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- Factorization and basic renormalizations

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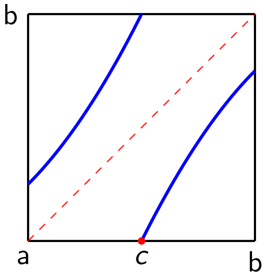

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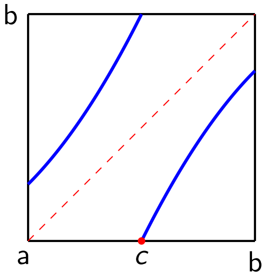

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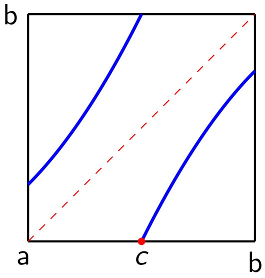

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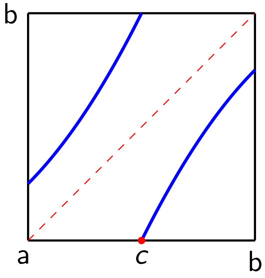

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Remark

The last condition implies that the set $\bigcup_{n\in\mathbb{N}_0} F^{-n}(c)$ is dense in $[a,b]$.

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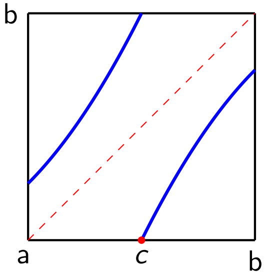

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We assume that $F(c) \in \{a,b\}$.

Motivation: Geometric models of Lorenz attractor

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- Poincaré maps in geometric models of Lorenz attractor.

V. S.Afraĭmovich, V. V. Bykov, L. P. Shil'nikov, *On attracting structurally unstable limit sets of Lorenz attractor type*. (in Russian) Trudy Moskov. Mat. Obshch. **44** (1982), 150–212.

J. Guckenheimer, *A strange, strange attractor*, in: J. E. Marsden and M. McCracken (eds.), The Hopf Bifurcation Theorem and its Applications, Springer, 1976, pp. 368–381.

R. F. Williams, *The structure of Lorenz attractors*. Inst. Hautes Ètudes Sci. Publ. Math. No. **50**, (1979), 73–99.

Motivation: Number theory

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- Expansions of real numbers in non-integer bases.

W. Parry, *On the β-expansions of real numbers*. Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

A. Rényi, *Representations for real numbers and their ergodic properties*. Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

Motivation: Neural dynamics

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- Dynamics of map-based neuron models.

P. Bartłomiejczyk, F. Llovera-Trujillo, J. Signerska-Rynkowska, *Analysis of dynamics of a map-based neuron model via Lorenz maps*. Chaos **34** (2024), 043110.

P. Bartłomiejczyk, F. Llovera Trujillo, J. Signerska-Rynkowska, *Spike patterns and chaos in a map-based neuron model*, Int. J. Appl. Math. Comput. Sci., **33** (2023), no.3, 395–408.

Standard doubling points construction: Phase space $\mathbb{X}$

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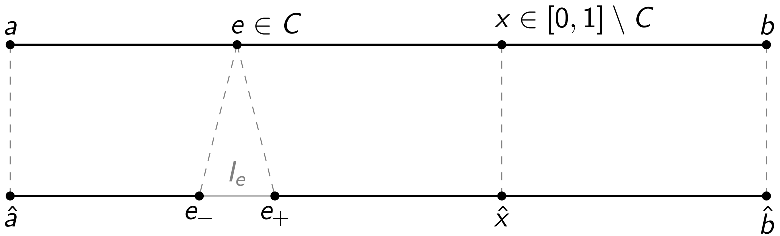
- Elements in $C := \left(\bigcup_{n=0}^{\infty} F^{-n}(\{c\})\right) \setminus \{a,b\}$ are doubled (we perform a kind of Denjoy extension).



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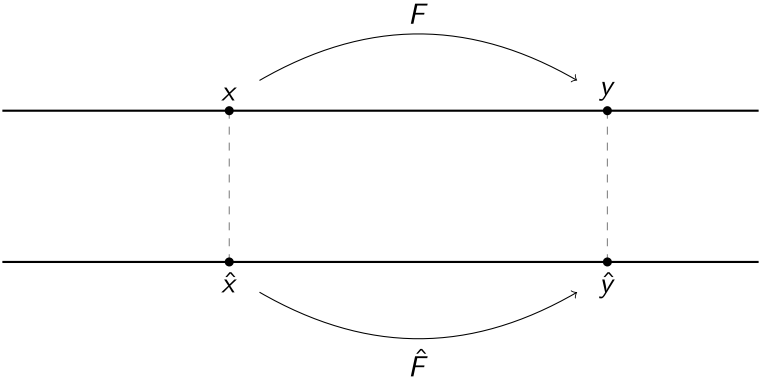

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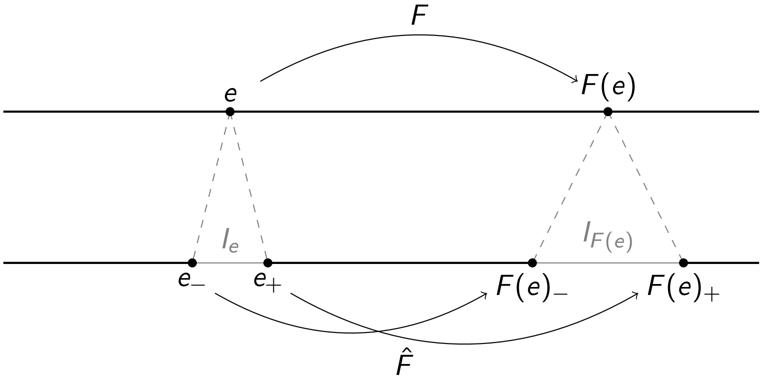

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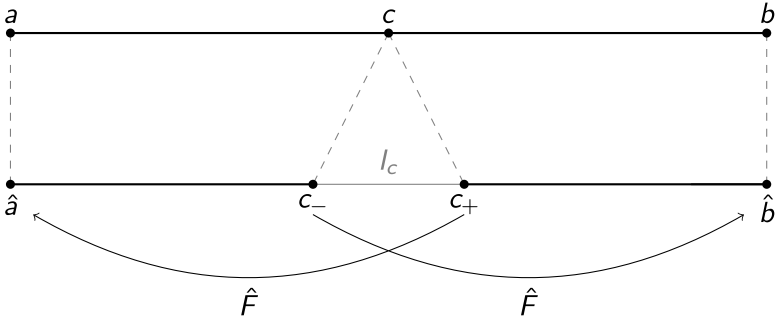

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- More details:

P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*. Israel J. Math. **80** (1992), 97–133.

Renormalization of Lorenz maps

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- Denote $p_i := T^i(0)$ and $q_i := T^i(1)$.

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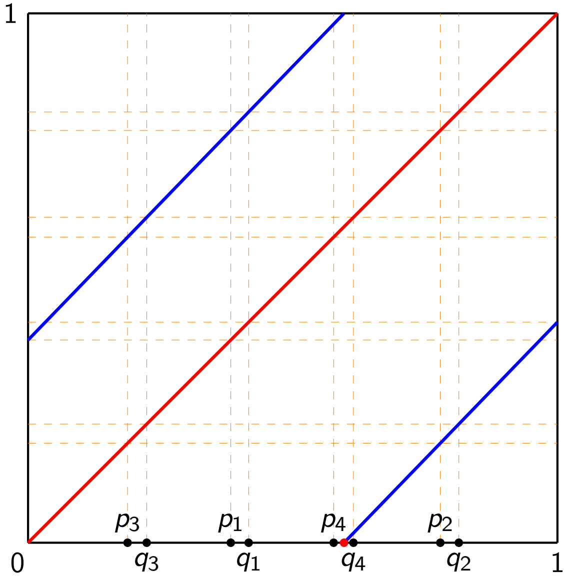

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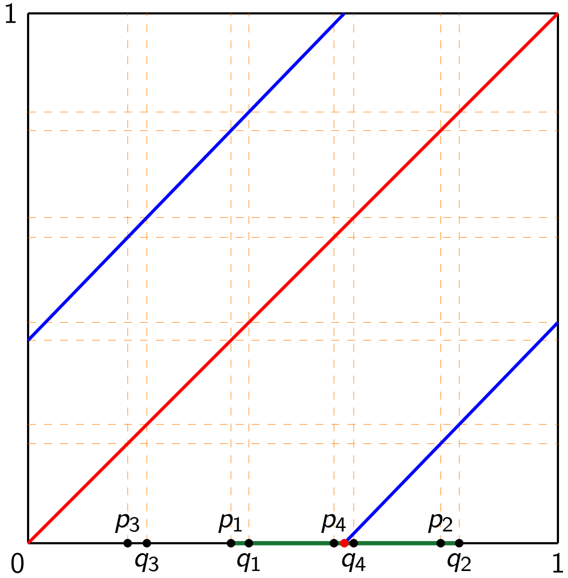

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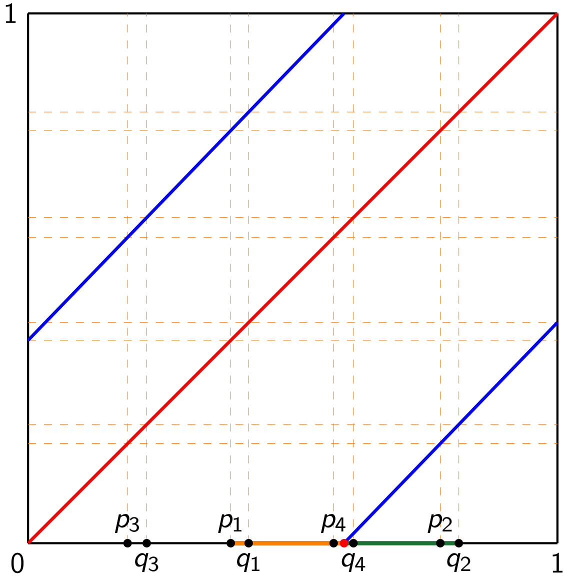

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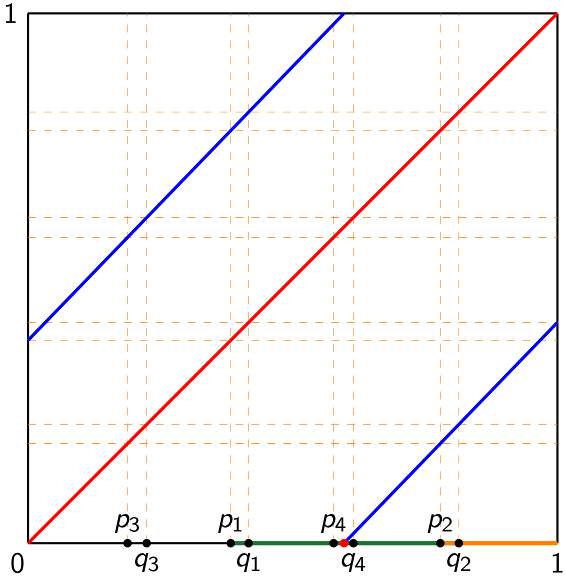

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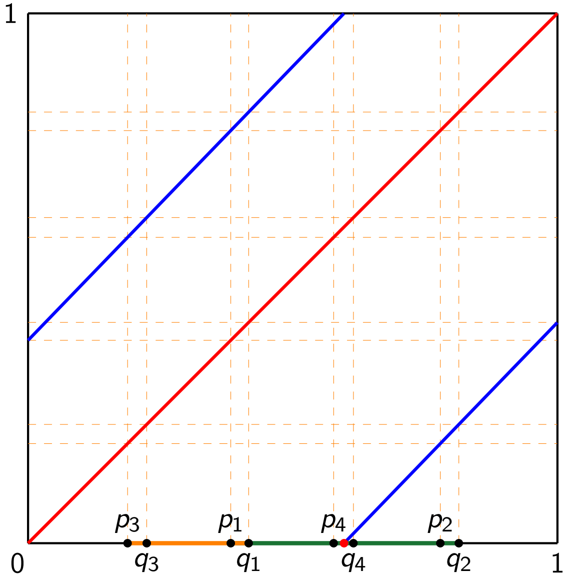

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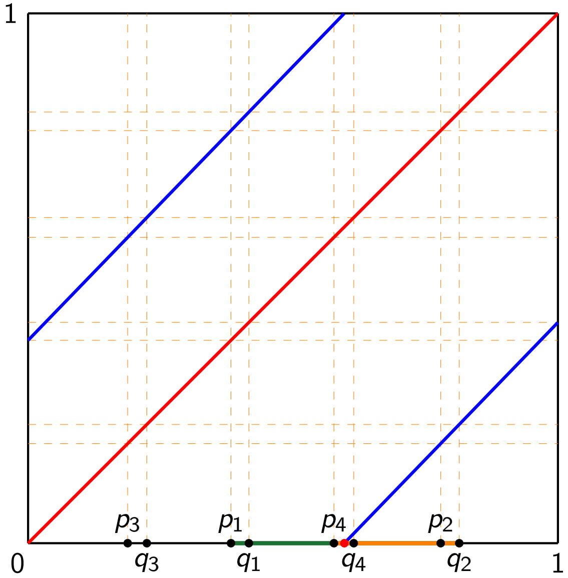

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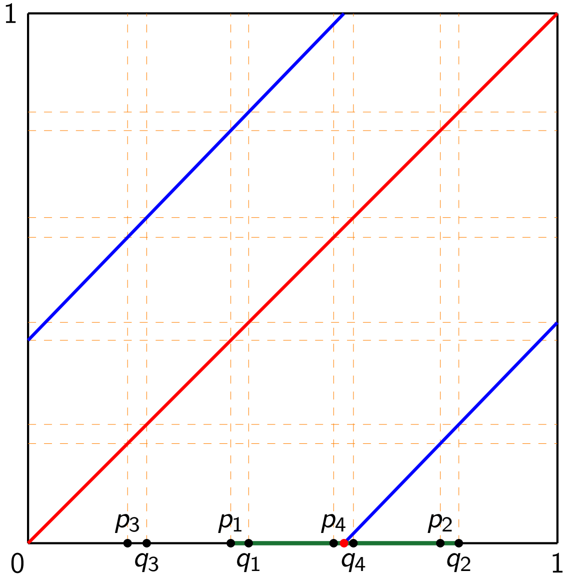

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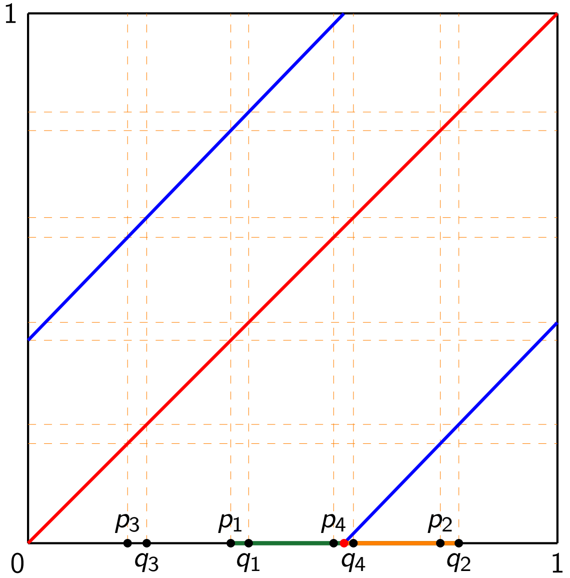

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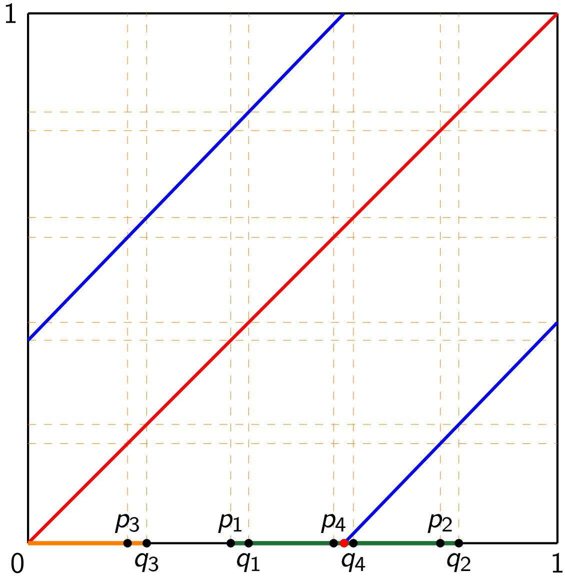

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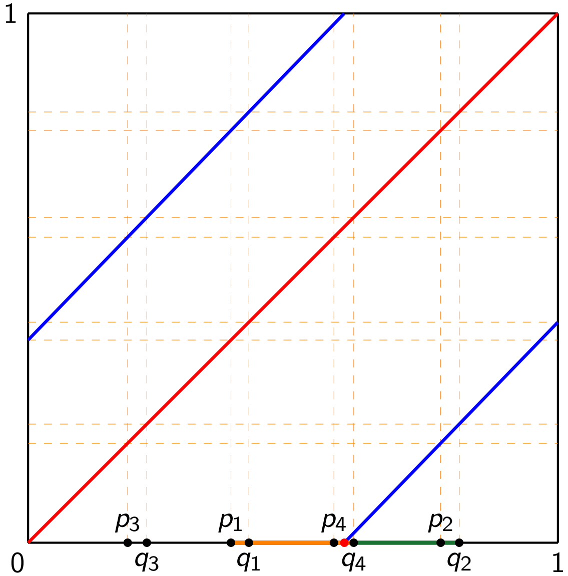

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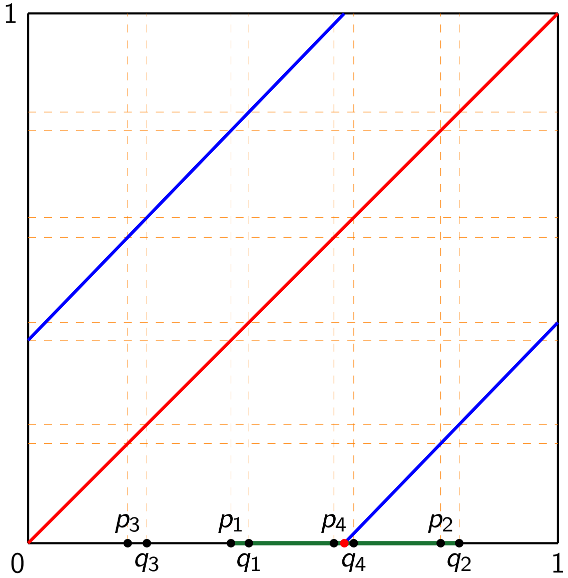

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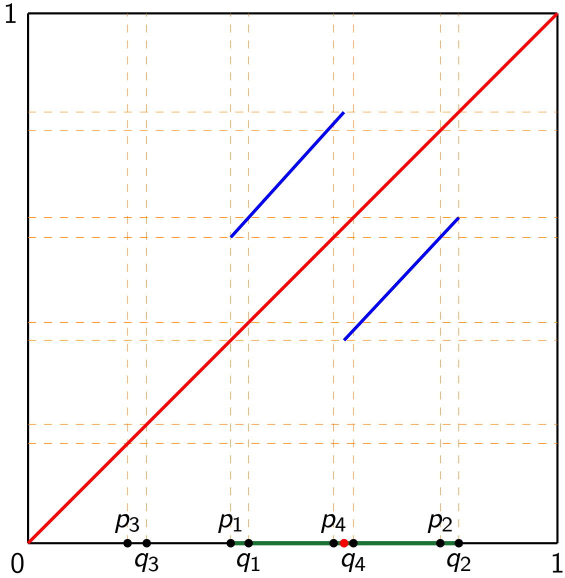

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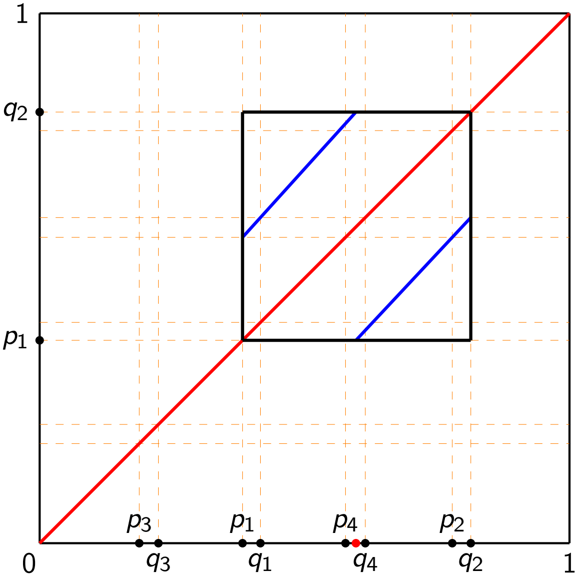

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Let $U \subset [a, b]$ be an open set. By $N(U)$ we denote the smallest integer $n \geqslant 0$ such that $c \in F^n(U)$.

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Then $F^l(e_-) = e_-$, $F^r(e_+) = e_+$ and the following map

$$R_E F(x) = \begin{cases} F^l(x), & x \in [F^{r-1}(a), c) \\ F^r(x), & x \in (c, F^{l-1}(b)] \end{cases}$$

is a renormalization of F.

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It may happen that the set J_G is empty or not completely invariant!

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Proposition

If there exists a proper completely invariant closed set E such that $R_E F = G$, then $E = J_G$.

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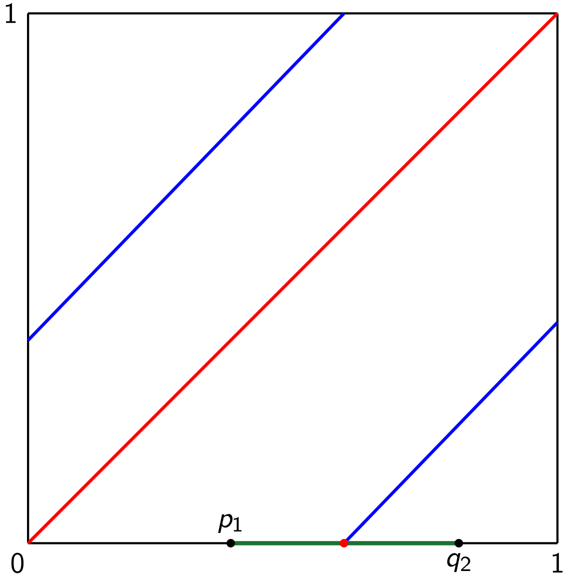

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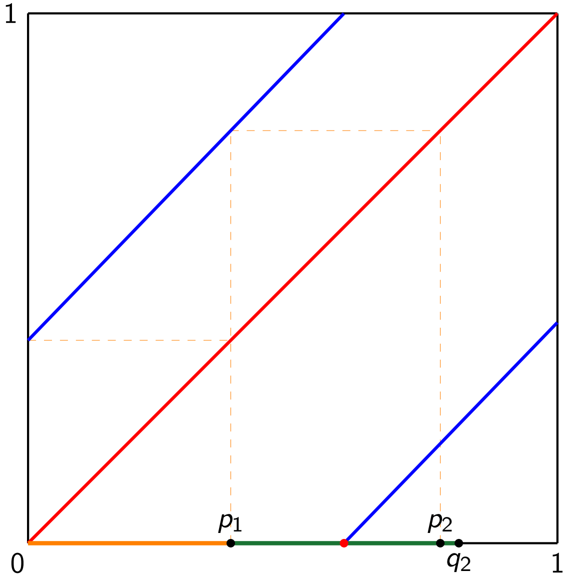

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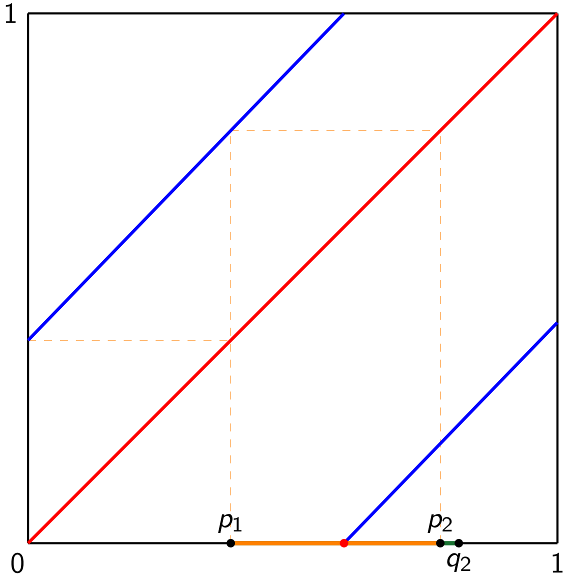

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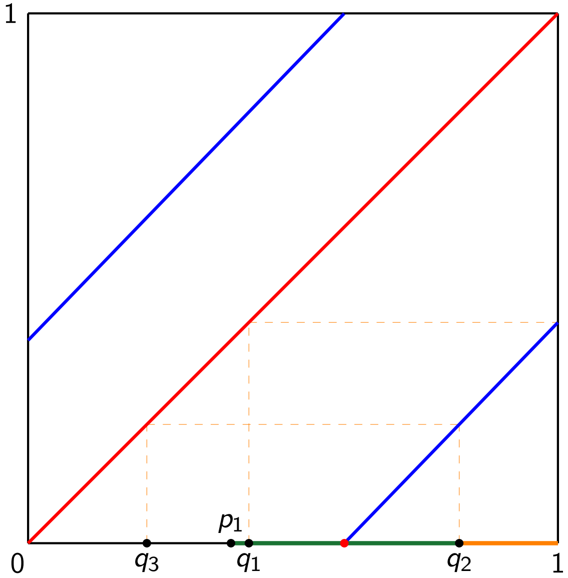

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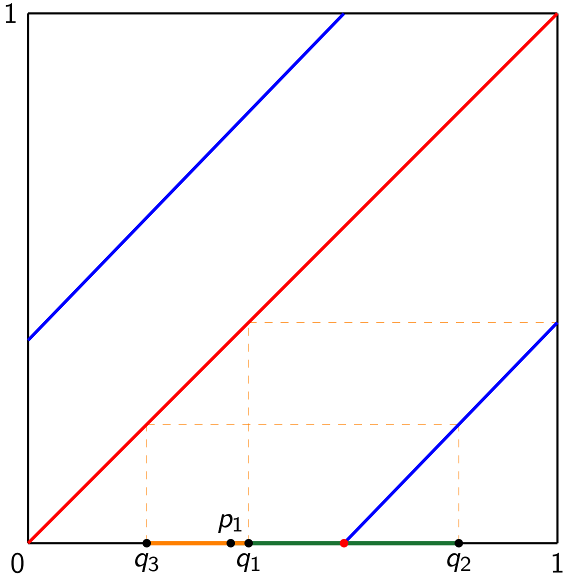

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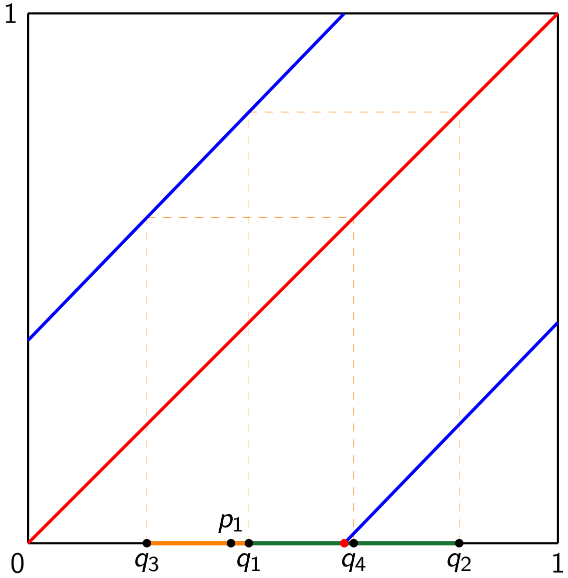

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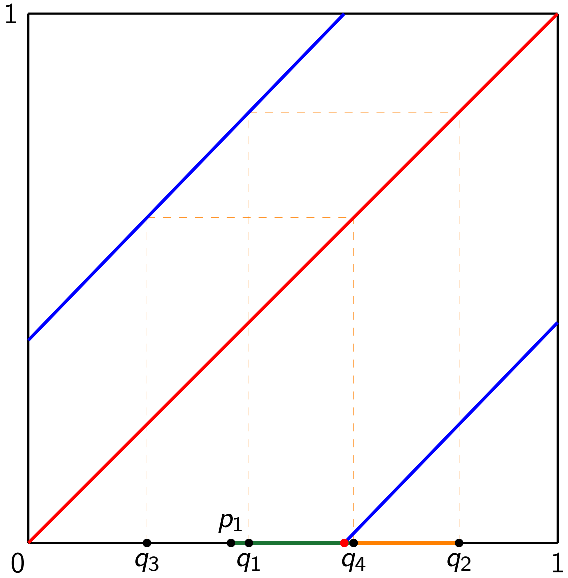

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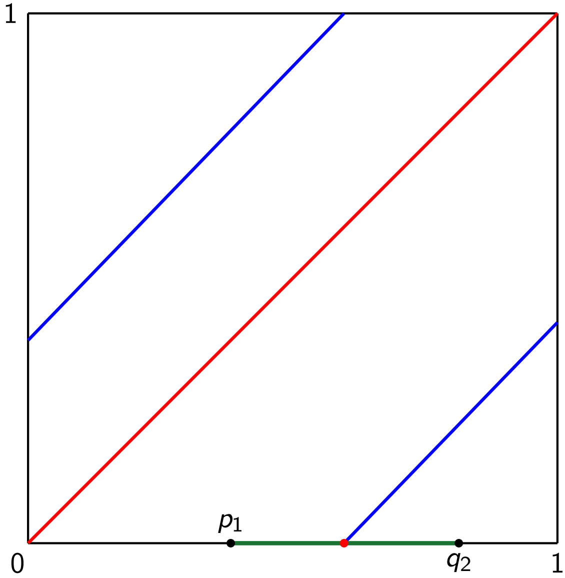

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- Recall

$$T\colon [0,1] \ni x \mapsto \beta x + \alpha \pmod 1 \in [0,1],$$

where

$$\beta := \frac{9\sqrt[5]{2}}{10} \approx 1.03383 \qquad \text{and} \qquad \alpha := \frac{\sqrt[5]{2}}{3} \approx 0.38289.$$

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Ding's Theorem

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Question

Which renormalizations can be obtained from the procedure described in the Ding's theorem? In particular, when the set J_G is nonempty and completely invariant?

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Let $F\colon [a,b] \to [a,b]$ *be an expanding Lorenz map.*

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is itself a Lorenz map (on the interval $[u,v]$*),*

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Remark

In the above definition we do not assume that the map G is proper renormalization, i.e. it may happen that $[u,v] = [a,b]$.

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2. *We have $F_n = (F_{n-1}^2, F_{n-1}^2)$. Then the set J_{F_n} contains a periodic orbit, $R_{J_{F_n}} F = F_n$ and $l_n = r_n$.*

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3. *We have $F_n \neq (F_{n-1}^2, F_{n-1}^2)$ and $F_i = (F_{i-1}^2, F_{i-1}^2)$ for some $i < n$. Then the set J_{F_n} contains a periodic orbit, but $R_{J_{F_n}} F \neq F_n$ and $l_n \neq r_n$. In fact $R_{J_{F_n}} F = F_j$, where $0 < j < n$ is the largest index for which the map F_j is neither trivial nor special trivial renormalization of F_{j-1}.*

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*The renormalizations (F, F^2), (F^2, F) and (F^2, F^2) of an expanding Lorenz map F are called **basic renormalizations**.*

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Corollary

Let F be an expanding Lorenz map and $G = (F^l, F^r)$ be a renormalization of F that can be factorized into basic renormalizations. Then the set J_G is nonempty and defines the renormalization G, i.e. $R_{J_G} F = G$, if and only if $l = r$.

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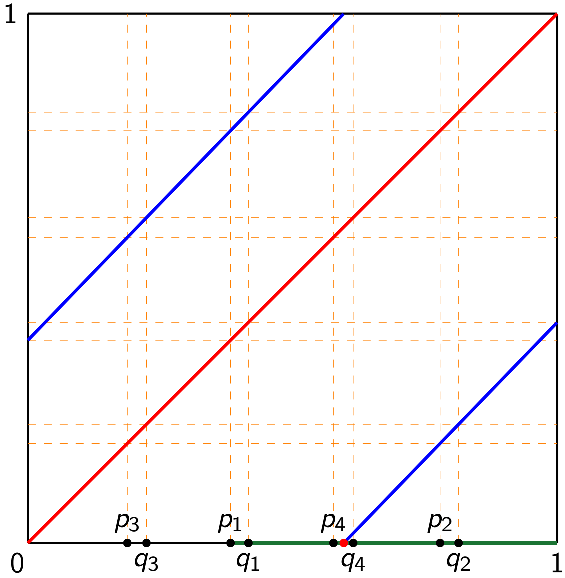

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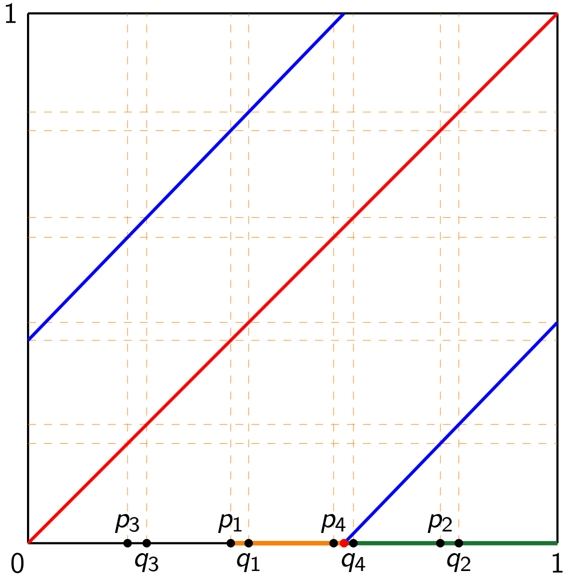

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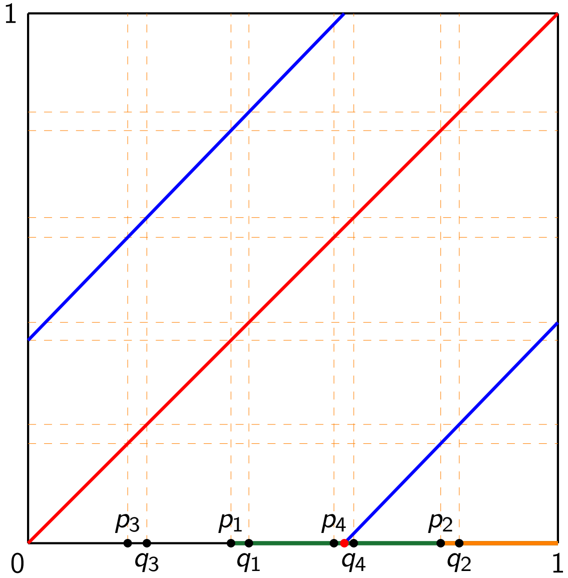

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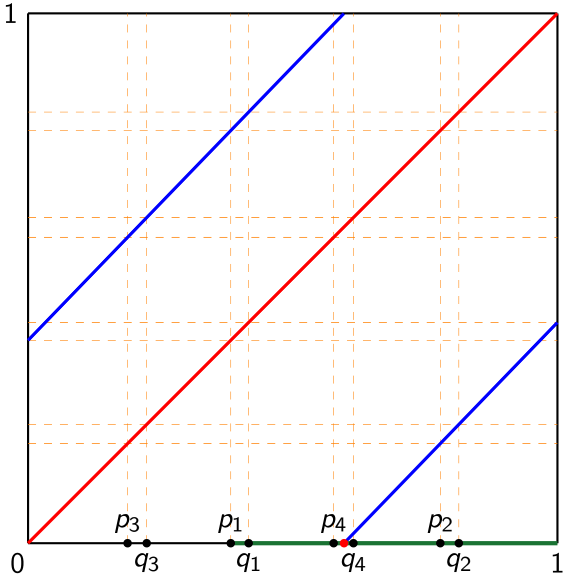

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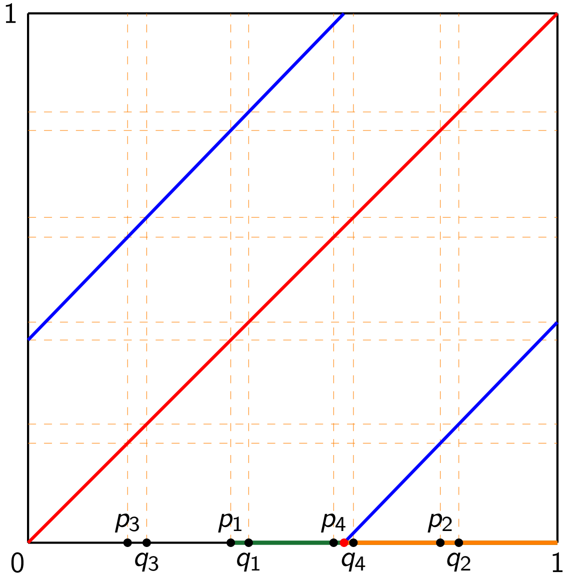

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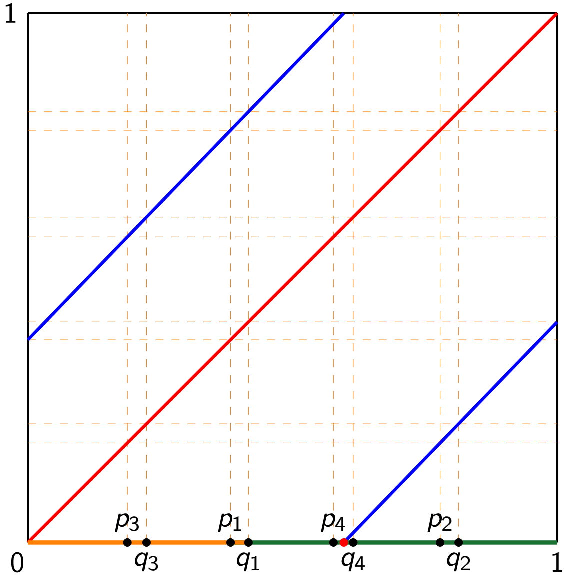

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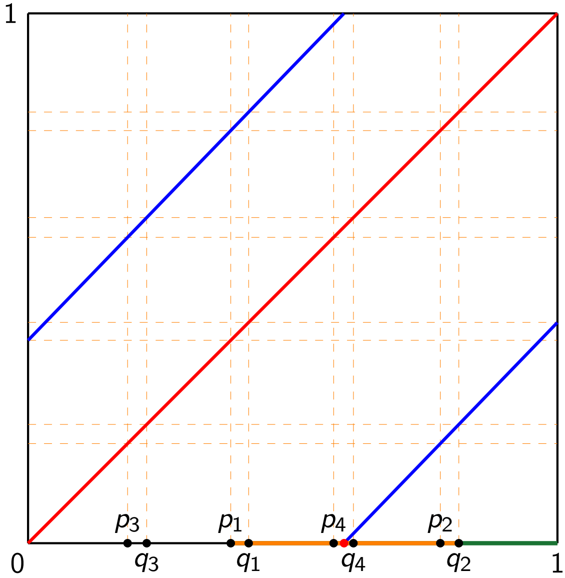

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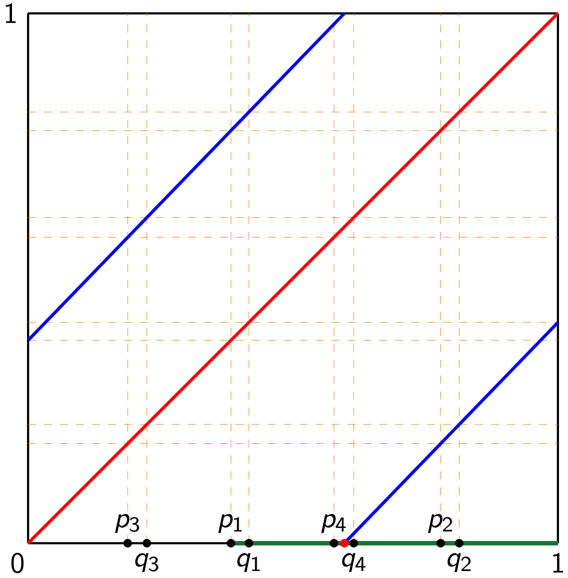

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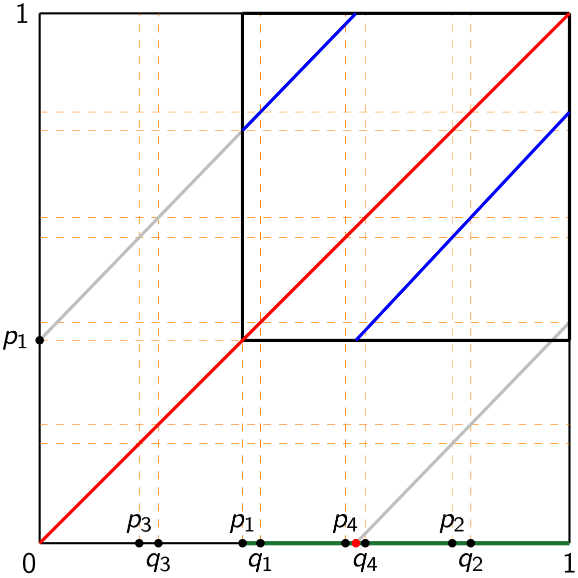

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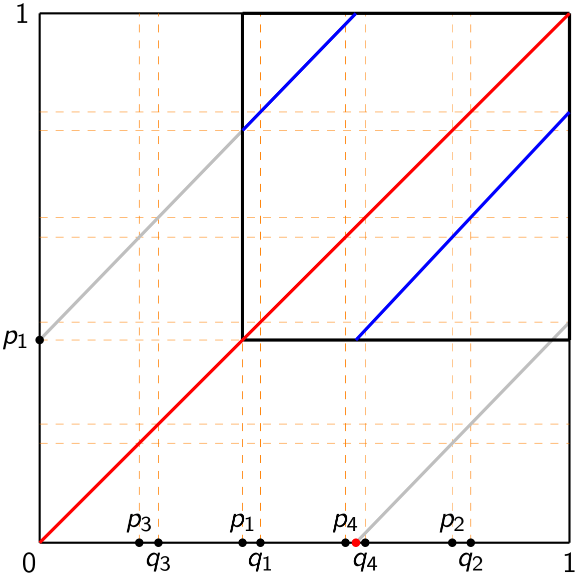

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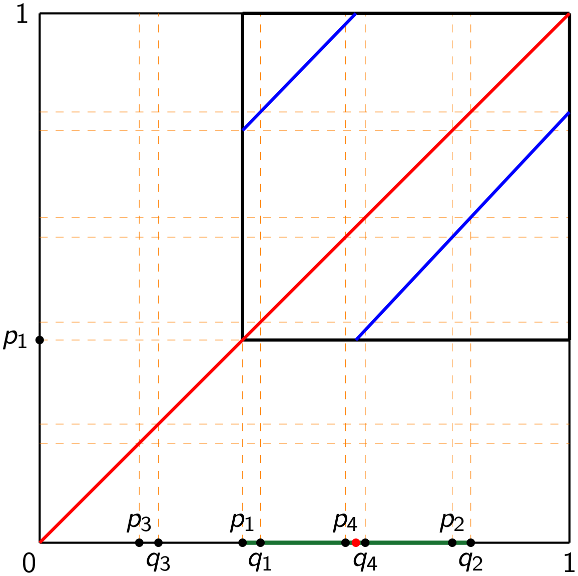

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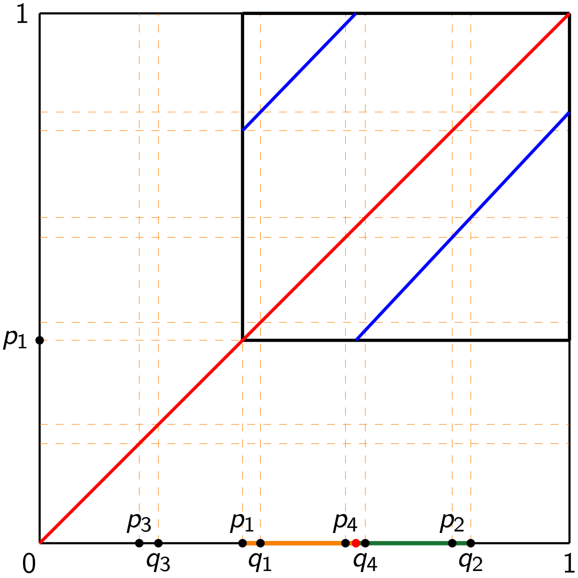

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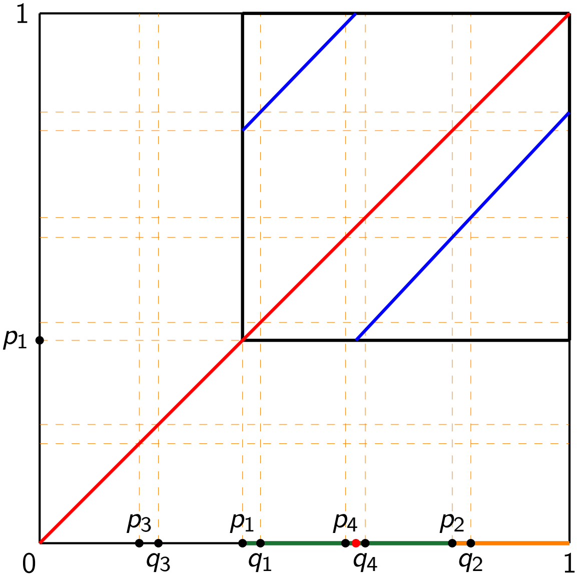

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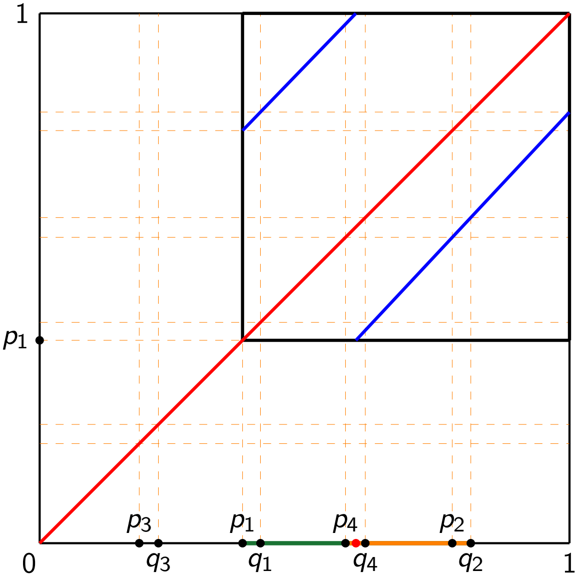

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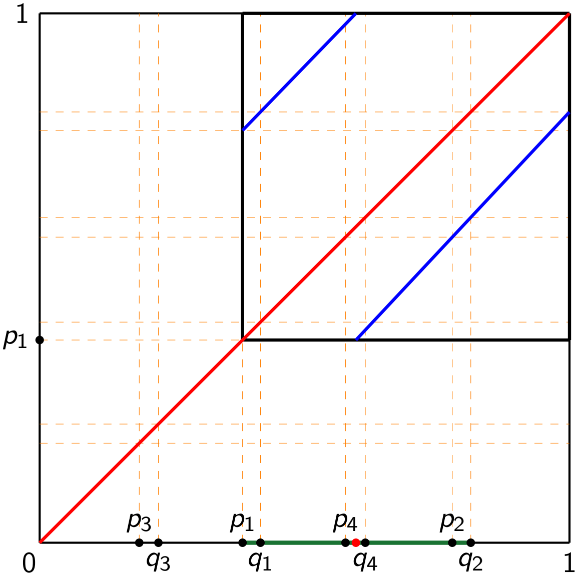

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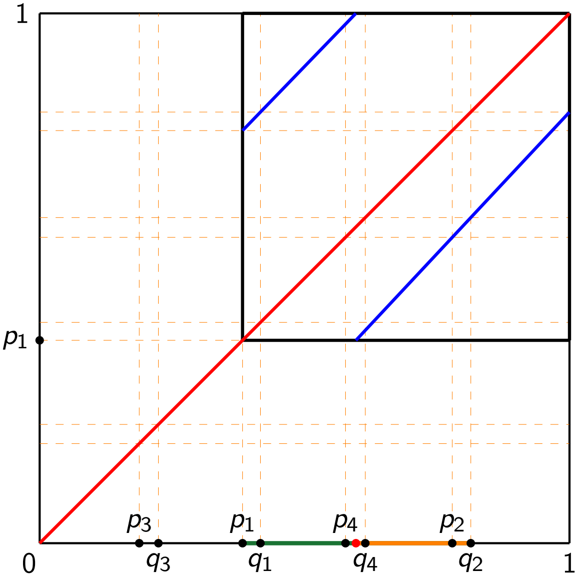

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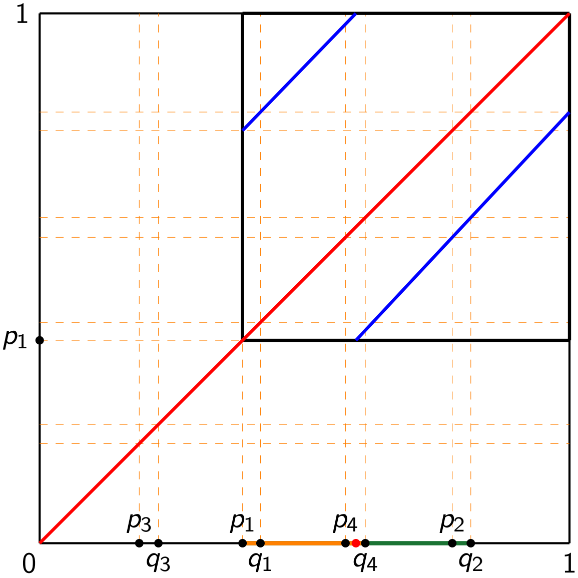

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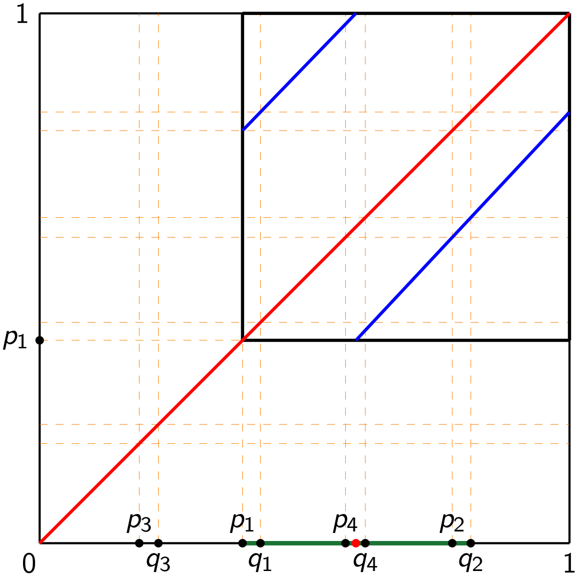

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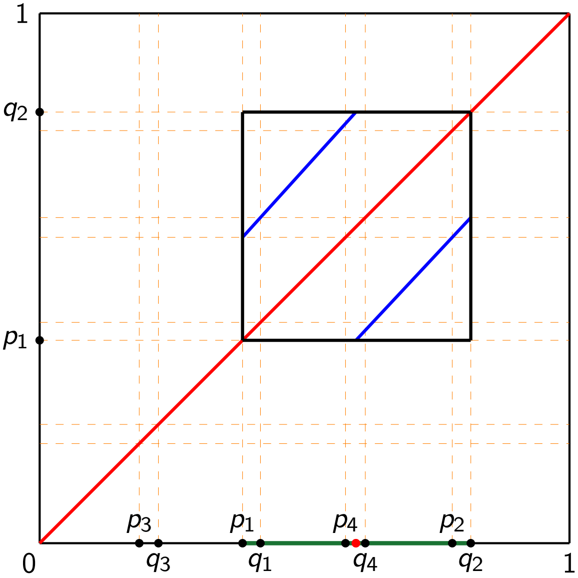

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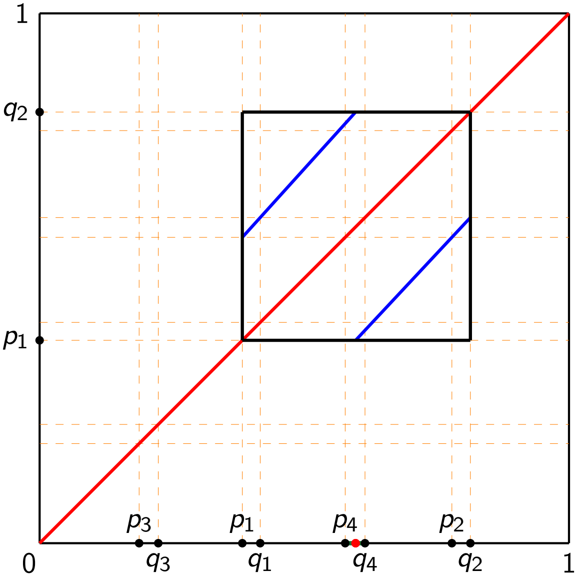

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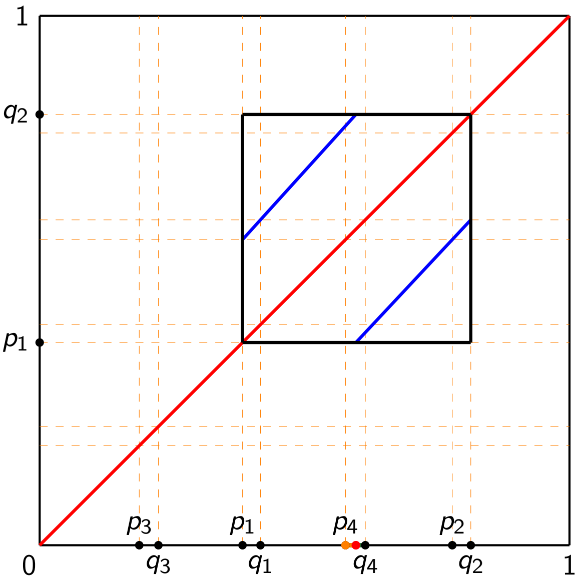

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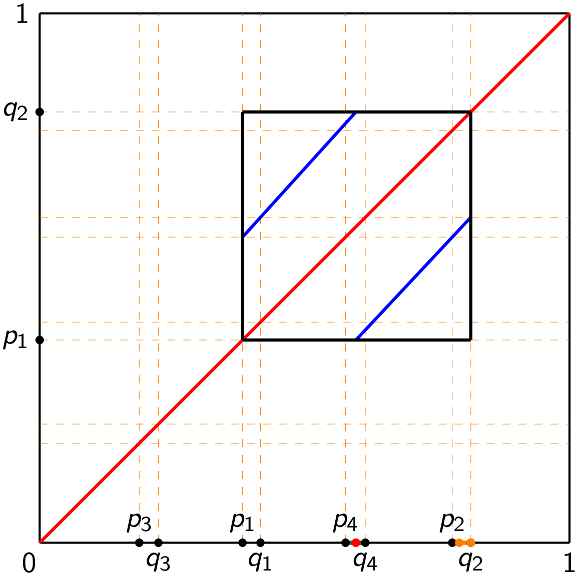

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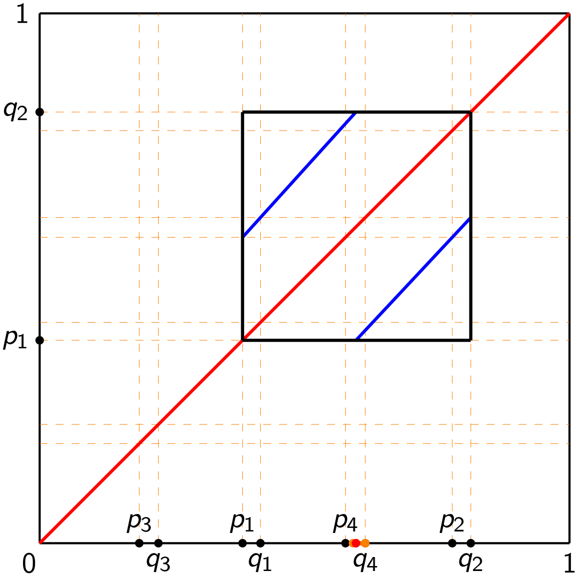

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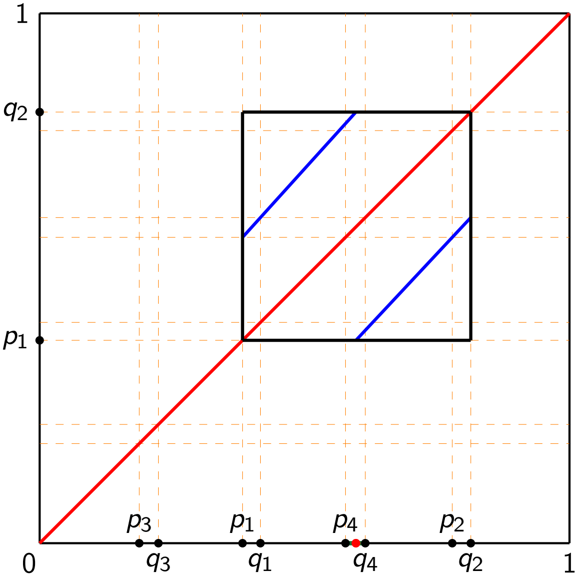

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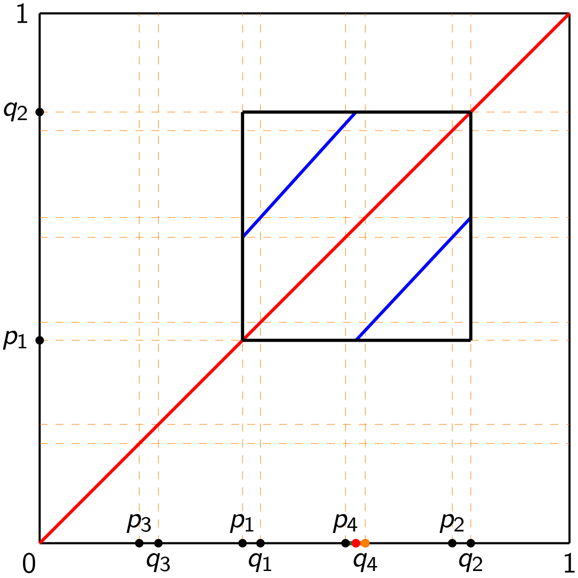

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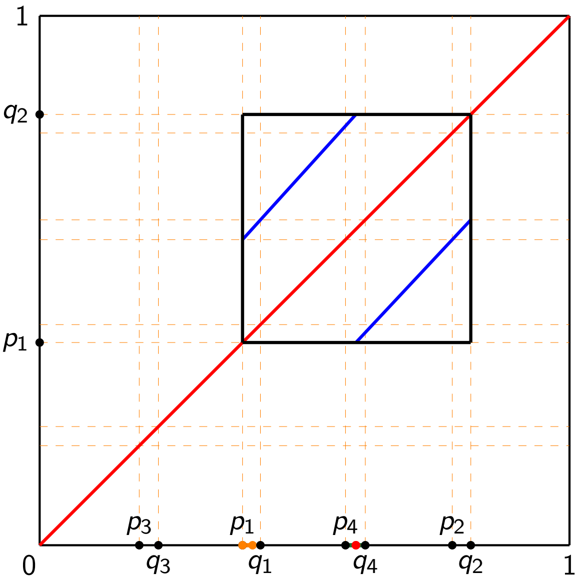

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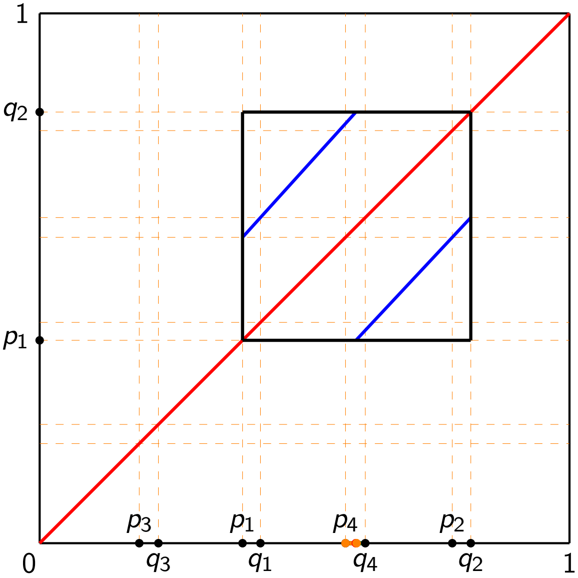

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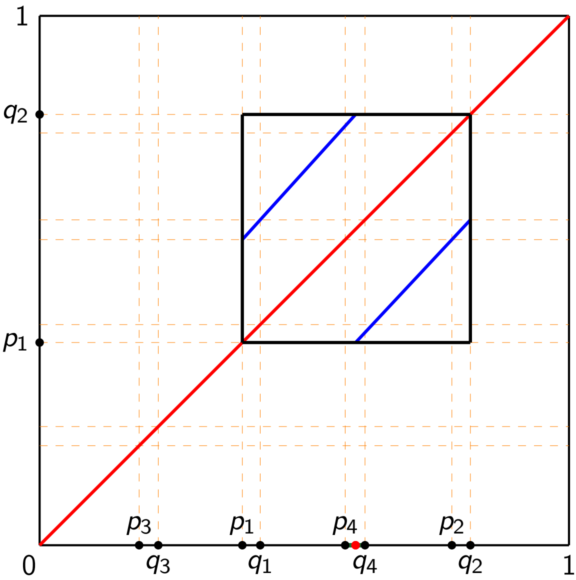
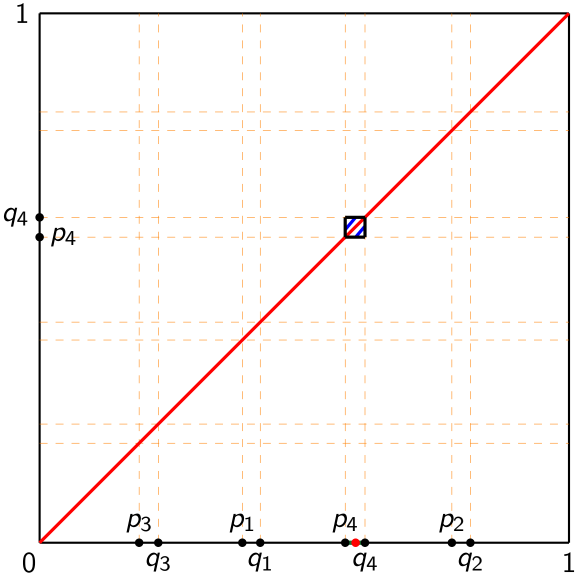
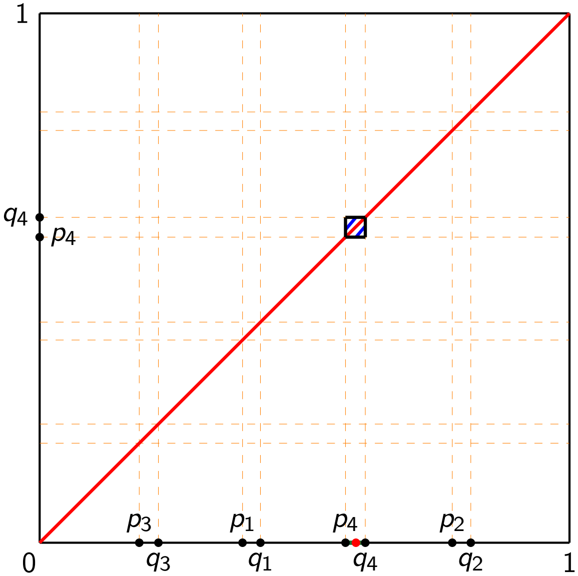


Example: Graph of the map $T_3 = (T_2^2, T_2^2) = (T^5, T^5)$



Example: Graph of the map $T_3 = (T_2^2, T_2^2) = (T^5, T^5)$



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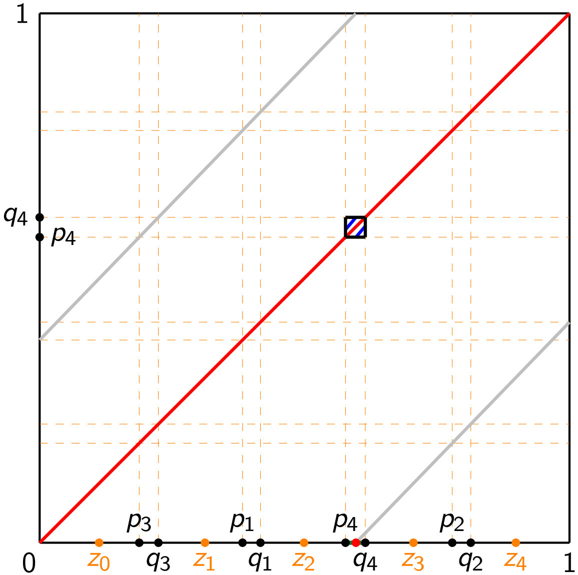

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- The renormalization $\tilde{G} = (T^5, T^5)$ is a composition of two trivial renormalizations T_1, T_2 and the renormalization $T_3 = (T_2^2, T_2^2)$.

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- The renormalization $\tilde{G} = (T^5, T^5)$ is a composition of two trivial renormalizations T_1, T_2 and the renormalization $T_3 = (T_2^2, T_2^2)$. Moreover $J_{\tilde{G}} \neq \emptyset$ and $R_{J_{\tilde{G}}} T = \tilde{G}$.

Basic renormalizations

Basic renormalizations

P. Glendinning, C. Sparrow, *Prime and renormalizable kneading invariants and the dynamics of expanding Lorenz maps*, Physica D, **62** (1993), 22–50.

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Question

What about the expanding Lorenz maps that do not have a constant slope? Is it possible to obtain a renormalization $G = (F^l, F^r)$ with $l \neq r$ using the procedure from the Ding's theorem?

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The end

Thank you for your attention!