Game comonads: logical and homotopical aspects

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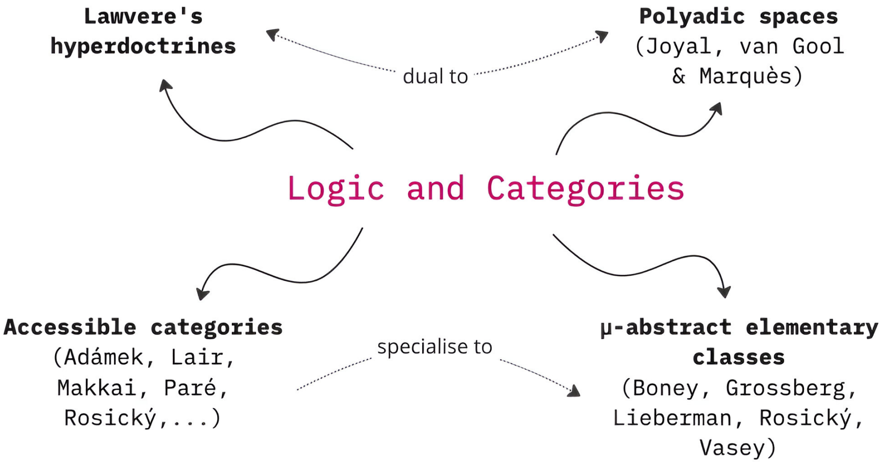
1. Logic, categories, and resources

2. Games: unravelling and covering

3. Homomorphism counting and Preservation theorems



These are powerful tools for studying (infinitary) extensions of first-order logic and (non-elementary) classes of mathematical structures in a syntax-free way, as well as theories in coherent, intuitionistic and continuous logics, etc.

Logical resources

In contrast, in this talk I will be interested in capturing fine structure “down below”, typically in resource-bounded fragments of first-order logic.

The idea of stratifying formulas by **logical resources**, typically represented as complexity measures of formulas such as quantifier rank or number of variables, is central to finite model theory and descriptive complexity.

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Provisos:

- We work with relational structures, but do not assume they are finite.
- Infinitary logic $\mathcal{L}_{\infty,\omega}$ (= first-order logic with infinite $\bigwedge$ and $\bigvee$) crops up.
- We want to avoid the use of the Compactness Theorem for first-order logic.

Life without compactness

Finite model theory $=$ Model theory $-$ Compactness

A proof that does not use compactness (ultraproducts, saturated extensions, etc.)

- is more likely to admit a relativisation to finite models
- typically carries quantitative information about the complexity of the objects
- often results in stronger conclusions (both for finite and infinite models).

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I will present some first steps towards an "axiomatic resource-sensitive model theory", based on categorical and comonadic methods.

Contributors to this programme include:
Samson Abramsky, Adam Ó Conghaile, Anuj Dawar, Tomáš Jakl, Thomas Laure, Dan Marsden, Yoàv Montacute, Tom Paine, Colin Riba, Nihil Shah and Pengming Wang.

Three examples (I)

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Using $\wedge, \vee, \exists$ but not $\neg, \forall$

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Finite HPT (Rossman, 2005)

A first-order sentence is preserved under homomorphisms between finite structures if, and only if, it is equivalent over finite structures to an existential positive one.

Three examples (II)

Modal logic: $p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\mathrm{ML} \xrightarrow{\text{standard translation}} \mathrm{FO}[x]$$

$$[\![p]\!]_x := P(x)$$
$$[\![\Box\varphi]\!]_x := \forall y.\, R(x,y) \to [\![\varphi]\!]_y$$
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Theorem (Rosen, 1997)

A first-order formula $\varphi(x)$ of quantifier rank $\leq k$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to a modal formula of modal depth $\leq 2^k$.

Maximum nesting of modalities $\Box, \Diamond$

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The same result holds relative to finite Kripke models.

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$\mathrm{FO}^n(\#)$	:	n-variable first-order logic with counting quantifiers

For each $i \in \mathbb{N}$, add $\exists_{\geq i}$

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Theorem (Dvořák, 2010)

The following statements are equivalent for all finite graphs G and H:

1. $G \equiv^{\mathrm{FO}^n(\#)} H$
2. $\mathbf{Gr}(F, G) \cong \mathbf{Gr}(F, H)$ for all finite graphs F of tree-width $< n$.

Games: the case of modal logic

In (finite) model theory, one is typically not interested in objects up to isomorphism, but only up to definable properties.

Games are a useful tool to establish whether two structures are equivalent with respect to a given logic fragment. Consider e.g. the case of modal logic:

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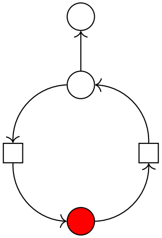
Bisimulation game for modal logic

Played by Spoiler (S) and Duplicator (D) on two pointed Kripke models (A, a) and (B, b).

Initial position: $(a_0, b_0) := (a, b)$. At round i, with current position (a_i, b_i), S picks one of the models, e.g. A, and $a_{i+1} \in A$ such that $a_i \, R^A \, a_{i+1}$. D responds with $b_{i+1} \in B$ such that $b_i \, R^B \, b_{i+1}$. If D has no such response available, they lose. D wins after k rounds if a_i and b_i satisfy the same unary predicates, for all i with $0 \leq i \leq k$.

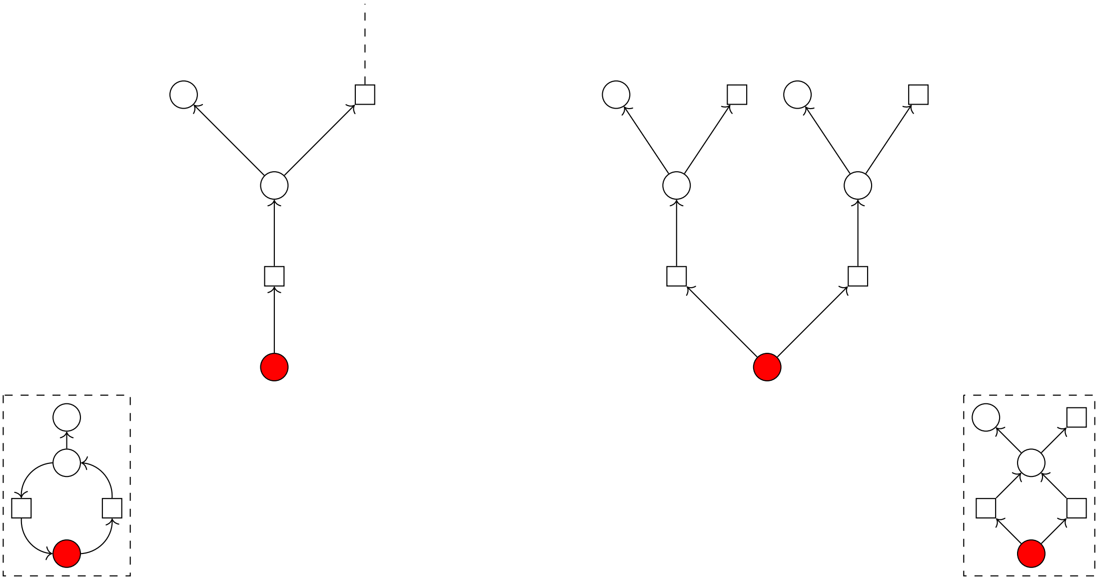
Theorem (Hennessy–Milner, 1980)

D has a winning strategy in the k-round bisimulation game if, and only if, (A, a) and (B, b) satisfy the same modal formulas of depth $\leq k$.



Duplicator wins the 3-round game but loses the 4-round game.
So, the least modal depth of a formula distinguishing the two models is 4.

Tree unravelling



Tree unravelling

The tree unravelling $R(A, a)$ of a Kripke model (A, a) is again a Kripke model, with distinguished element (a)—the one-element sequence. Moreover, it satisfies:

$(\star)$ $\forall x \in R(A, a)$, there is a unique path from the distinguished element of $R(A, a)$ to x.

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$(\star)$ $\forall x \in R(A, a)$, there is a unique path from the distinguished element of $R(A, a)$ to x.

Let $\mathbf{K}$ be the category of Kripke models, and $\mathbf{S}$ its full subcategory defined by the objects satisfying $(\star)$. The tree unravelling exhibits $\mathbf{S}$ as a coreflective subcategory of $\mathbf{K}$:

$$\mathbf{S} \underset{\hookrightarrow}{\overset{R}{\underset{\top}{\longleftarrow}}} \mathbf{K}$$

The objects of $\mathbf{S}$ are called synchronization trees, as they carry a "definable" tree order, namely the reflexive transitive closure of their Kripke relation.

Similarly, for each positive integer n, the tree unravelling to depth n defines a coreflection $R_n \colon \mathbf{K} \to \mathbf{S}_n$ of $\mathbf{K}$ onto the full subcategory $\mathbf{S}_n$ of synchronization trees of height at most n.

Comonads

Coreflective subcategories induce idempotent comonads. In particular, tree unravellings induce idempotent comonads on $\mathbf{K}$.

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Consider the functor $T\colon \mathbf{Set} \to \mathbf{Set}$ sending X to the set of non-empty finite words on X.

Monad structure on T	Comonad structure on T
unit $\eta_X\colon X \to TX$, $x \mapsto (x)$	**counit** $\varepsilon_X\colon TX \to X$, $(x_1, \ldots, x_n) \mapsto x_n$
multiplication $\mu_X\colon T^2X \to TX$, $((x, y), (z)) \mapsto (x, y, z)$	**comultiplication** $\delta_X\colon TX \to T^2X$, $(x_1, \ldots, x_n) \mapsto ((x_1), (x_1, x_2), \ldots, (x_1, \ldots, x_n))$
Algebras: semigroups	**Coalgebras**: forest orders

A comonad (T, ε, δ) is **idempotent** if δ is a natural isomorphism.

The comonads we consider are liftings of the comonad of forests to relational structures.

From unravellings to coverings

Recently it has emerged that various notions of games can be encoded by means of **comonads** on the category of structures [Abramsky, Dawar & Wang, 2017].

Crucially, the latter are *not* idempotent in general. This means that, unlike in the modal case, the forest order on coalgebras is not definable.

Given such a game comonad G, we can think of GA as a "covering" of the structure A.

$$GA \xrightarrow{\varepsilon_A} A$$

The Ehrenfeucht-Fraïssé comonad

Ehrenfeucht-Fraïssé (EF) game

Played by S and D on two structures A and B. Adjust the bisimulation game as follows:
No initial position, and S and D are not required to move along any “accessibility relation”.
D wins after k rounds if the relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial isomorphism.

Theorem (Ehrenfeucht & Fraïssé, 1950s)

D has a winning strategy in the k-round Ehrenfeucht-Fraïssé game if, and only if, A and B satisfy the same sentences of quantifier rank $\leq k$.

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Let **R** be the category of (relational) structures and their homomorphisms. For each $k \geq 1$, we define an EF comonad $\mathbb{E}_k$ on **R**.

- The universe of $\mathbb{E}_k A$ is $\bigcup_{i=1}^{k} A^i$, the set of plays in A of length at most k.

  If R is a (say, binary) relation, $R^{\mathbb{E}_k A}$ consists of the pairs of sequences (s, t) such that: s and t are comparable in the prefix order, and $(\mathtt{last}(s), \mathtt{last}(t)) \in R^A$.
- The counit is $\mathtt{last}_A \colon \mathbb{E}_k A \to A$, $[a_1, \ldots, a_m] \mapsto a_m$.
- The comultiplication $\mathbb{E}_k A \to \mathbb{E}_k^2 A$ is $[a_1, \ldots, a_m] \mapsto [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_m]]$.

The Ehrenfeucht-Fraïssé comonad

Coalgebras for the EF comonads capture the combinatorial parameter of tree-depth:

Theorem (Abramsky & Shah, 2018)

$A \in \mathbf{R}$ has tree-depth $\leq k$ if, and only if, it admits a coalgebra structure $\alpha \colon A \to \mathbb{E}_k A$.

Coalgebras for $\mathbb{E}_k$ can be described as objects of $\mathbf{R}$ equipped with a compatible forest order of height $\leq k$. Coalgebra morphisms are homomorphisms that are also forest morphisms.

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Preservation of logic fragments. Denote by FO_k first-order logic with quantifier rank $\leq k$. In the category of coalgebras for $\mathbb{E}_k$,

- the homomorphism preorder captures preservation of $\exists^+\mathrm{FO}_k$-sentences
- the isomorphism relation captures equivalence in $\mathrm{FO}_k(\#)$

Other fragments of FO_k (existential, positive, etc.) can be captured in a similar fashion. E.g., equivalence in FO_k corresponds to the existence of a span of open maps.

Categories from games

The ideas just outlined are not specific to EF or bisimulation games and have been extended to a number of other model comparison games.

Game comonads	Logical resources	Combinatorial parameters
pebble comonad	number of variables	tree-width
Ehrenfeucht–Fraïssé comonad	quantifier rank	tree-depth
modal comonad	modal depth	synchronization-tree depth
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Starting from concrete notions of games, we build a (resource-indexed) family of comonads and consider the associated categories of coalgebras.

Can we recognise the comonads arising from games, and their categories of coalgebras? We can attempt to isolate the fundamental properties of these categories; this leads to an **axiomatic perspective** on game comonads.

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Definition

1. An object P of **C** is a path provided its poset $\mathbb{S}\,P$ of $\mathcal{M}$-subobjects is a finite chain.
2. A path embedding in **C** is an $\mathcal{M}$-morphism $P \rightarrowtail X$ whose domain is a path.

For any $X \in$ **C**, the sub-poset $\mathbb{P}\,X$ of $\mathbb{S}\,X$ consisting of the path embeddings is a tree. Further, $\mathbb{P}\,X$ is non-empty if the factorisation system is stable and **C** has an initial object.

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This minimal amount of structure on $\mathbf{C}$ allows us to define various abstract notions of games between objects X, Y by playing on the associated trees $\mathbb{P}\,X, \mathbb{P}\,Y$.

Back-and-forth game in C. Played by S and D on objects X and Y. This is essentially the bisimulation game played on the trees $\mathbb{P}\,X$ and $\mathbb{P}\,Y$ with initial position given by the roots. The accessibility relation is the immediate-successor relation, and at each round D must ensure that the selected path embeddings have isomorphic domains.

Games from categories

The ensuing notion of back-and-forth equivalence on objects of $\mathbf{C}$ can be transferred to other categories $\mathbf{E}$ via adjunctions:

$$\mathbf{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbf{E} \quad (L \dashv R) \tag{1}$$

For all $a, b \in \mathbf{E}$, define $a \leftrightarrow^R b$ iff Ra and Rb are back-and-forth equivalent in $\mathbf{C}$.

All concrete examples of game comonads fit in this framework: $\mathbf{E} = \mathbf{R}$ is the category of relational structures, and $\mathbf{C}$ is the category of coalgebras for the comonad. The abstract game in $\mathbf{C}$ coincides with the corresponding concrete game (e.g., EF or bisimulation games).

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In general, preservation of resource-bounded logic fragments is captured by transferring along the adjunction $\mathbf{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{E}$ natural relations between objects of **C** (back-and-forth equivalence, homomorphism preorder, isomorphism, ...).

Arboreal categories

In the examples, the categories of coalgebras for game comonads satisfy additional properties. Most important of all, the full subcategory of paths $\mathbf{C}_p \hookrightarrow \mathbf{C}$ is dense.

These extra properties lead to the notion of arboreal category [Abramsky & LR, 2021].

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In arboreal categories, back-and-forth equivalence coincides with bisimilarity in the sense of [Joyal, Nielsen & Winskel, 1993]. Let us say that a morphism $X \to Y$ in an arboreal category $\mathbf{C}$ is open if it has the right lifting property wrt morphisms between paths.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Q & \longrightarrow & Y \end{array}$$

(Tightly related to the concept of open maps in presheaf toposes [Joyal & Moerdijk, 1994])

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E.g., bisimilarity in the category of coalgebras for $\mathbb{E}_k$ captures equivalence in FO_k.

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Homomorphism counting revisited

Theorem (Lovász, 1967)

Finite relational structures $A, B \in \mathbf{R}$ *are isomorphic iff* $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ *for all finite* $C \in \mathbf{R}$.

Lovász' result was generalised to locally finite categories with appropriate factorisation systems in [Pultr, 1973] and [Isbell, 1991].

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Hom-counting results in finite model theory were established in [Dawar, Jakl & LR, 2021] by:

(i) proving an abstract Lovász-type result for a certain class of categories;
(ii) instantiating this result for categories of coalgebras for game comonads.

Lovász' theorem (and its categorical generalisations) relies on a combinatorial argument combined with the fact that any left-cancellative finite monoid is a group.
In the non-locally-finite case, cf. [LR, 2022], we use:

Lemma (Numakura, 1952)

A compact Hausdorff topological monoid with the left-cancellation property is a group.

Homomorphism preservation theorems revisited

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The latter statement can be phrased, for any arboreal adjunction $L\colon \mathbf{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{E} : R$, as

(HP) For any full subcategory $\mathbf{E}'$ of $\mathbf{E}$ saturated under $\leftrightarrow^R$, $\mathbf{E}'$ is closed in $\mathbf{E}$ under morphisms precisely when it is upwards closed with respect to $\to^R$.

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Proposition

Property (HP) *holds if the arboreal adjunction has the bisimilar companion property, i.e.*

$$\text{for all } a \in \mathbf{E}, \quad a \leftrightarrow^R Ga \quad (\text{where } G := LR).$$

This yields equi-resource homomorphism preservation theorems for (graded) modal logic and guarded logic [Abramsky & LR, 2024]. These results relativise to finite structures because the corresponding comonads restrict to finite structures.

Homomorphism preservation theorems revisited

When an arboreal adjunction does *not* have the bisimilar companion property, we can try to "force" it using an upgrading argument:

We construct extensions $a^*, (Ga)^*$ of a and Ga, respectively, such that $a^* \leftrightarrow^R (Ga)^*$.

$$
\begin{array}{ccc}
a^* & \leftrightarrow^R & (Ga)^* \\
\uparrow & & \uparrow \\
a & \leftrightarrows^R & Ga
\end{array}
$$

This can be done assuming further properties of the arboreal adjunction, and leads to an axiomatic proof of the Equirank HPT.

Homomorphism preservation theorems revisited

When an arboreal adjunction does *not* have the bisimilar companion property, we can try to “force” it using an upgrading argument:

We construct extensions $a^*, (Ga)^*$ of a and Ga, respectively, such that $a^* \leftrightarrow^R (Ga)^*$.

$$\begin{array}{ccc} a^* & \leftrightarrow^R & (Ga)^* \\ \uparrow & & \uparrow \\ a & \leftrightarrows^R & Ga \end{array}$$

This can be done assuming further properties of the arboreal adjunction, and leads to an axiomatic proof of the Equirank HPT.

The extensions a^* and $(Ga)^*$ are constructed via a small object argument, similar to those in categorical homotopy.

Can we understand these upgrading arguments, which are pervasive in (finite) model theory, as instances of fibrant replacements in some (Quillen) model category?

Modal logic, presheaves and homotopy

Theorem (van Benthem, 199?; Rosen, 1997)

A modal formula of depth $\leq k$ is preserved under embeddings of Kripke models iff it is equivalent to an existential modal formula of depth $\leq k$ (constructed from the atoms and their negations, using $\wedge, \vee$ and $\Diamond$). Further, the result relativises to finite Kripke models.

Modal logic, presheaves and homotopy

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Recall that $\mathbf{S}_n$ is the full subcategory of $\mathbf{K}$ defined by synchronization trees of height $\leq n$. Let $\mathbf{P}_n$ be the subcategory of $\mathbf{S}_n$ consisting of traces and embeddings between them.

The presheaf category $\widehat{\mathbf{P}_n}$ admits a Quillen model structure in which

- cofibrations capture embeddings between synchronization trees;
- trivial fibrations capture p-morphisms.

The upgrading argument which gives the van Benthem–Rosen theorem amounts to an appropriate (cofibration, trivial fibration)-factorisation of arrows in $\widehat{\mathbf{P}_n}$ [LR, 2024].

Outlook

What I haven't mentioned:

- Other game comonads (Abramsky, Ó Conghaile, Dawar, Marsden, Montacute, Shah)
- Composition methods in finite model theory (Jakl, Marsden, Shah)
- Combinatorial parameters and density comonads (Abramsky, Jakl, Paine)
- Axiomatic characterisations of existential & positive fragments (Abramsky, Laure, LR)
- . . .

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Future directions:

- New examples: applications of arboreal adjunctions beyond (finite) model theory
- Translations between logic fragments: the category of arboreal categories
- Develop a homotopy theory of logical resources

Game comonads: logical and homotopical aspects

Survey article available at `arXiv:2407.00606`

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