Completely distributive lattices: Girard quantales, linear orders, complete congruences

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Plan

From discrete to continuous paths

Completely distributive lattices as continuous domains

Complete congruences of completely distributive lattices

The frame of quasi-regular open subsets

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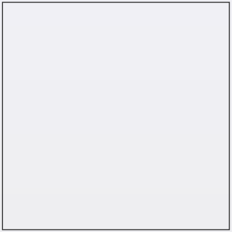

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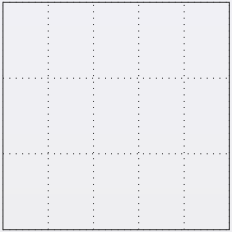

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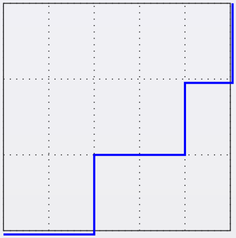

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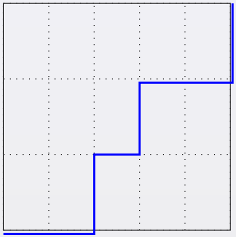

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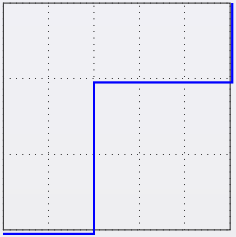

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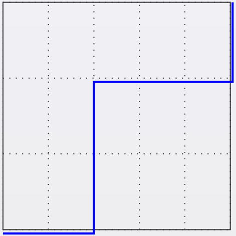

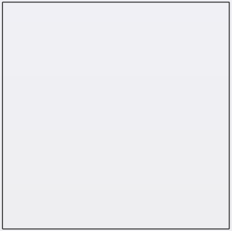

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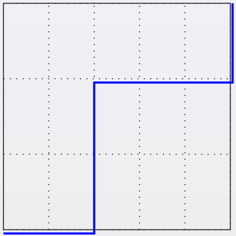

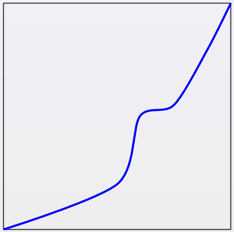

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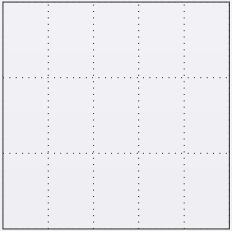

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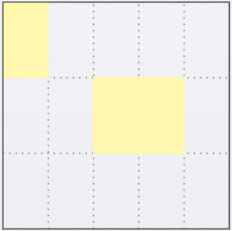

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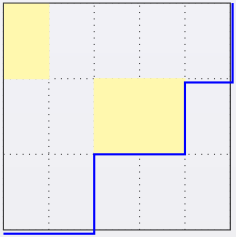

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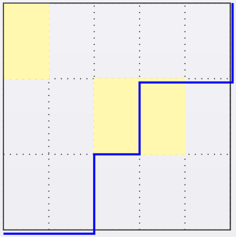

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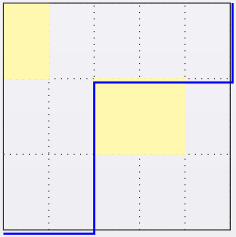

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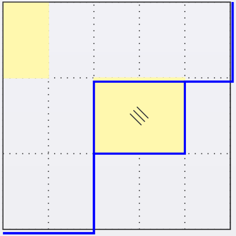

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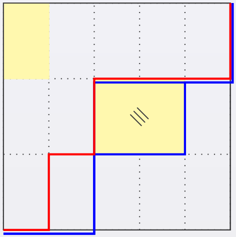

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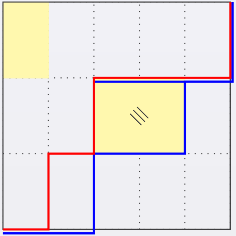

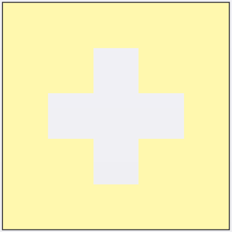

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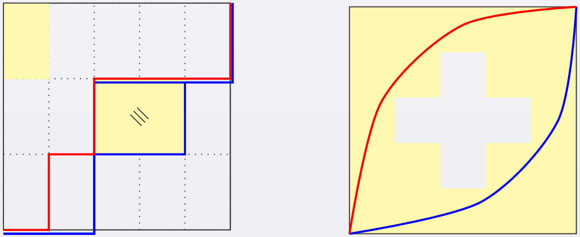

Similar to directed homotopies (Grandis 2009) as used in the modelling of concurrent computation (Fajstrup et al. 2016).

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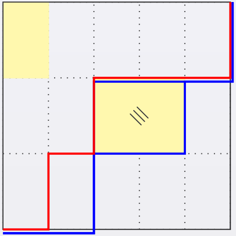

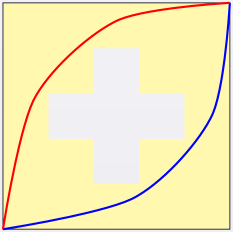

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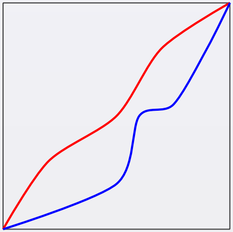
Problem: What about congruences of $Q_{\vee}(\mathbb{I})$?

The ordering on $Q_{\vee}(\mathbb{I})$

Say that $\pi_1 \leq \pi_2$ if π_2 is always on the left and above π_1:



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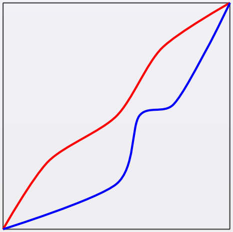

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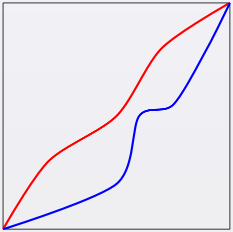

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Indeed $Q_\vee(\mathbb{I}) = \mathrm{Sup}(\mathbb{I}, \mathbb{I})$, with $\mathbb{I} = [0, 1]$.

Algebraic (and categorical) structures of $Q_\vee(\mathbb{I})$

Proposition. $Q_\vee(\mathbb{I})$ is:

- a quantale (where $\otimes$ is composition),
- a Girard quantale (where $(-)^*$ is reflection along diagonal),
- a completely distributive lattice.

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A *completely distributive lattice* is a complete lattice for which

$$\bigwedge_i \bigvee_j x_{i,j} = \bigvee_{f:I\to J} \bigwedge_i x_{i,f(i)} \,.$$

Every completely distributive lattice is a complete Heyting algebra, that is, a *frame*.

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Definition. P is a *continuous domain* if it is a domain and, for each $x \in P$,

- $\twoheaddownarrow x := \{ y \in P \mid y \ll x \}$ is directed,
- $x = \bigvee \twoheaddownarrow x$.

Examples

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$$\mathcal{J}(\, Q_{\vee}(\mathbb{I})\,) \simeq [0,1)^{op} \times (0,1]\,,$$

$$(x,y) \ll (x',y') \quad \text{iff} \quad x' < x \quad \text{and} \quad y < y'\,.$$

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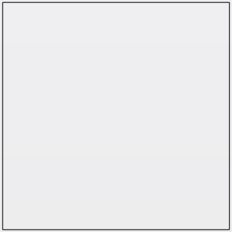

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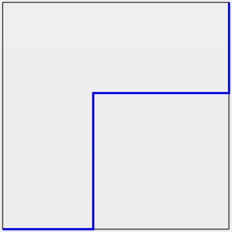

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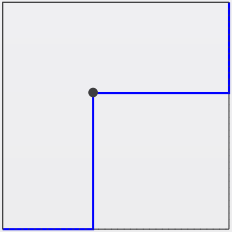

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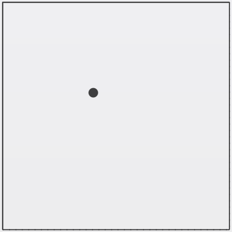

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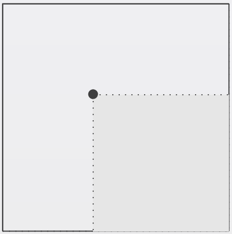

Completely distributive lattice as continuous domains

Proposition. See e.g. (Gierz et al. 2003) A complete lattice L is completely distributive if and only if

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Lemma. A completely distributive lattice is a *spatial* and *cospatial* frame.

Spatial : it is generated under infima by its meet-prime elements.

Cospatial : it is generated under suprema by its join-prime elements.

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Theorem (Lawson 1979; Hoffmann 1981). These correspondences make up a dual equivalence of categories.

See e.g. (Gehrke and van Gool 2024).

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- By frame duality, a quotient of a completely distributive lattice L to a (co)spatial (co)frame yields a subset $S \subseteq J(L)$ that is closed under directed suprema.
- In many cases—for $Q_\vee(\mathbb{I})$, in particular—this is a bijection between frame congruences and subsets closed under directed suprema.
  But not always.

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Definition. A point $s \in S$ is *exterior in* S if, for some $x \in P$,

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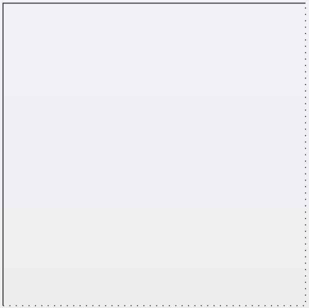
Remark. P is endowed with the topology where

$$\text{closed} = \text{closed under directed suprema.}$$

This is not the Scott-topology, it is a variant.

By the previous slide, we shall most often assume that $S \subseteq P$ is closed.

Examples



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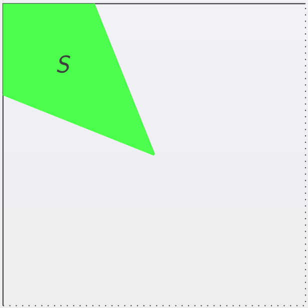

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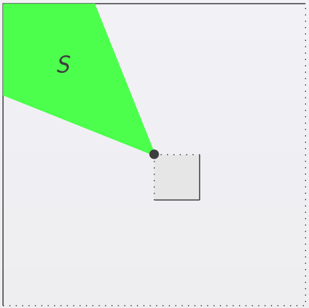

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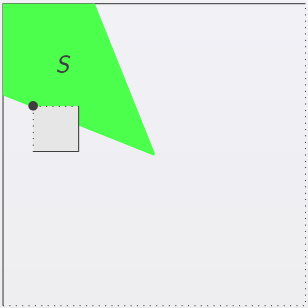

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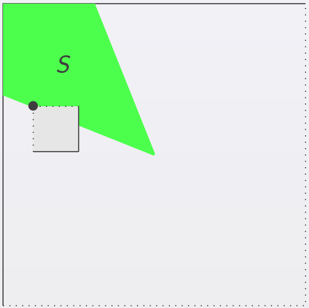


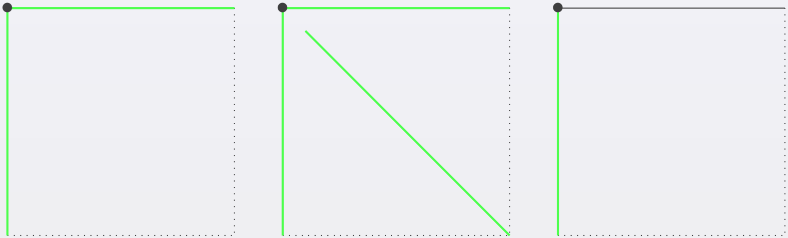
Every exterior point of S belongs to the frontier of S, but not conversely.

Quasi-regular closed subsets and complete congruences

Theorem. (Calk and Santocanale 2024) For $S \subseteq P$ closed, TFAE:

- S is has no exterior points.
- $\{ x \in S \mid x \ll s \}$ is cofinal in $\{ y \mid y \ll s \}$, for each $s \in S$.
- S is a continuous domain,
  and the inclusion $\iota : \mathsf{S} \hookrightarrow \mathsf{P}$ preserves the $\ll$ relation.
- The map
  $$\iota^{-1} : \mathcal{D}(P) \longrightarrow \mathcal{D}(S), \qquad X \mapsto X \cap S$$
  (is surjective and) preserves arbitrary (infima and) suprema.
  That is, it is a *complete* map (and consequently $D(S)$ is a completely distributive lattice).
- The congruence $\equiv_S$ is complete.

Examples



From the left to the right:

(i) S is not a continuous domain: $\Downarrow p = \emptyset$,

(ii) S is not a continuous domain: $\bigvee \Downarrow p < p$,

(iii) the inclusion of S into P does not preserve $\ll$.

Complete maps: a refinement of Hoffmann-Lawson duality

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Theorem. (Calk and Santocanale 2024) Let $f : P_1 \longrightarrow P_2$ be a Scott-continuous map between continuous domains. TFAE:

1. $f^{-1} : \mathcal{D}(P_2) \longrightarrow \mathcal{D}(P_1)$ is complete (i.e. preserves arbitrary suprema).
2. f preserves the way-below relation.

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2. f preserves the way-below relation.

NB: the above statement can be recovered from the literature (Hofmann and Stralka 1976), modulo some key remarks.

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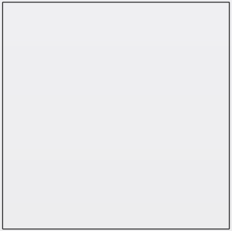

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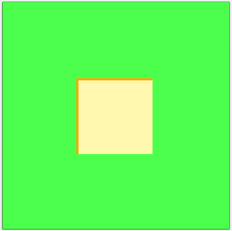

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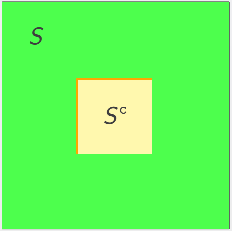

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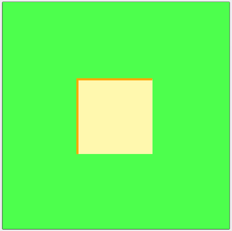

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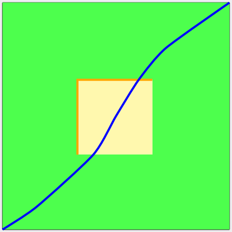

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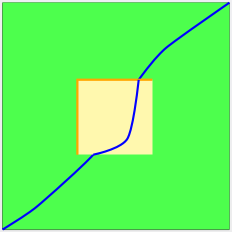

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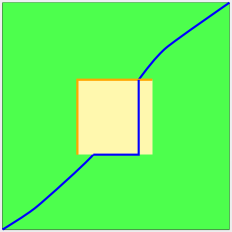

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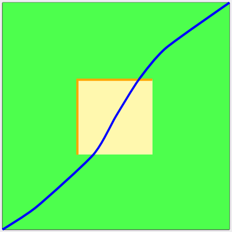

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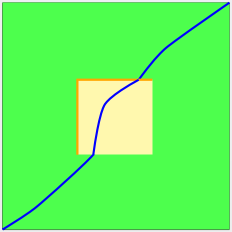

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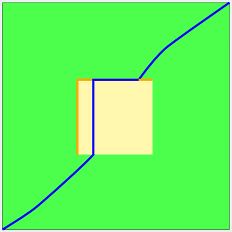

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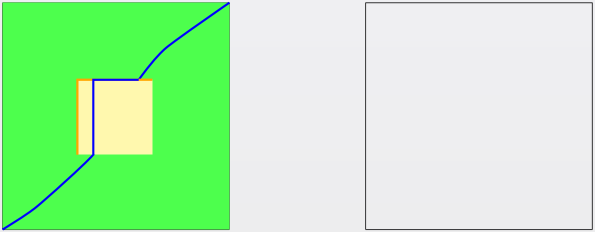

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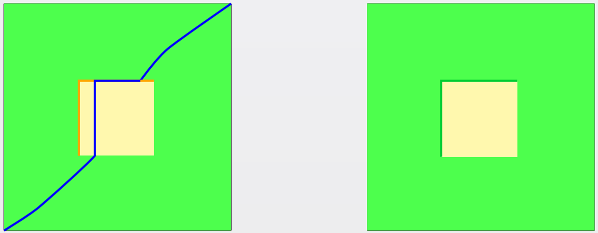

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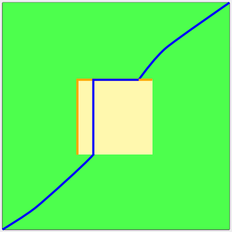

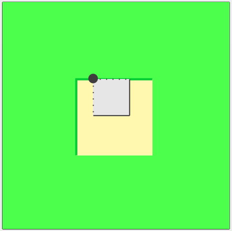

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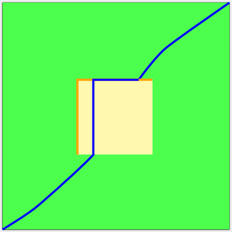

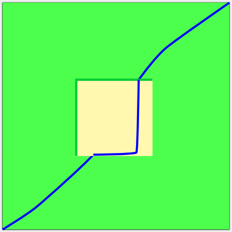

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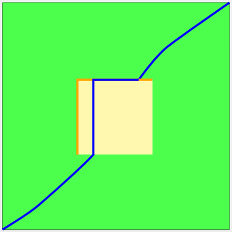

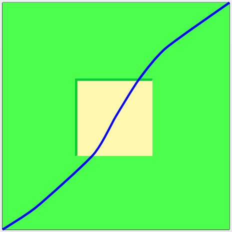

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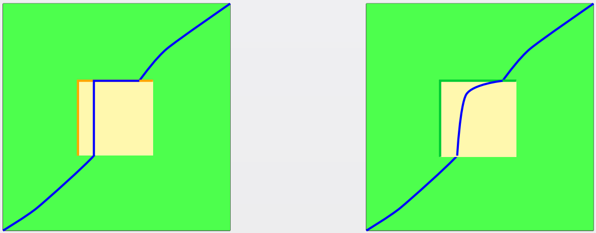

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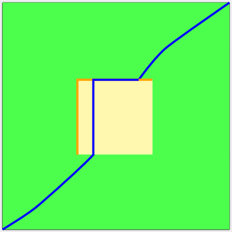

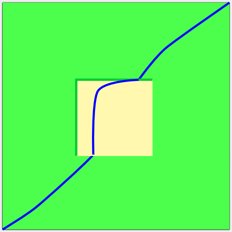

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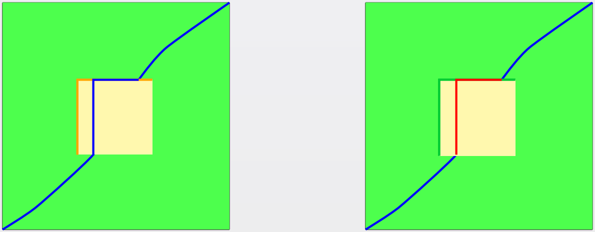

Plan

From discrete to continuous paths

Completely distributive lattices as continuous domains

Complete congruences of completely distributive lattices

The frame of quasi-regular open subsets

Complete congruences make up a (dual) frame

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Corollary. (Calk and Santocanale 2024) The poset of quasi-regular open subsets of P is a frame.

The poset of complete congruences of a completely distributive lattice is a dual frame.

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Thanks for your attention

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