Quantale-valued maps and partial maps

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Partial maps

A partial map $f \colon X \longrightarrow Y$ between sets is a map from a (possibly empty) subset of X to Y. Let

$$\mathbf{Set}^{\partial}$$

denote the category of sets and partial maps. The following results are well known:

- $\mathbf{Set}^{\partial}$ is equivalent to the category $\mathbf{Set}_*$ of pointed sets and basepoint-preserving maps.
- $\mathbf{Set}_*$ is isomorphic to the coslice category $\{\star\}/\mathbf{Set}$.

The maybe monad

The forgetful functor

$$\mathsf{U}\colon \mathbf{Set}_* \longrightarrow \mathbf{Set}$$

admits a left adjoint, which carries a set X to

$$X_+ := X \amalg \{\star\},$$

giving rise to the maybe monad on $\mathbf{Set}$, whose Eilenberg-Moore category and Kleisli category are

$$\{\star\}/\mathbf{Set}(\cong \mathbf{Set}_*) \quad \text{and} \quad \mathbf{Set}^{\partial},$$

respectively, which are equivalent. In particular, $*/\mathbf{Set}$, $\mathbf{Set}_*$ and $\mathbf{Set}^{\partial}$ are all monadic over $\mathbf{Set}$.

Quantales

Throughout, let

$$(\mathsf{Q}, \&, k)$$

denote a non-trivial, commutative and unital quantale; that is,

- Q is complete lattice, and
- $(\mathsf{Q}, \&, k)$ is a commutative monoid,

such that

$$\bot < k \quad \text{and} \quad p \,\&\, \Big(\bigvee_{i \in I} q_i\Big) = \bigvee_{i \in I} p \,\&\, q_i$$

for all $p, q_i \in \mathsf{Q}\,(i \in I)$. The right adjoint induced by $\&$ is denoted by $\to$, which satisfies

$$p \,\&\, q \leqslant r \iff p \leqslant q \to r$$

for all $p, q, r \in \mathsf{Q}$.

Q-relations

A Q-relation $\varphi\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ between sets is a function

$$\varphi\colon X \times Y \longrightarrow \mathsf{Q}.$$

Sets and Q-relations constitute a quantaloid

$$\mathsf{Q}\text{-}\mathbf{Rel}.$$

The identity Q-relation on a set X is given by

$$\mathrm{id}\colon X \longrightarrow\!\!\!\!\!\!\mapsto X, \quad \mathrm{id}_X(x,y) = \begin{cases} k & \text{if } x = y, \\ \bot & \text{else.} \end{cases}$$

Quantaloids

A quantaloid $\mathcal{Q}$ is a category with a class of objects $\mathcal{Q}_0$ in which $\mathcal{Q}(p,q)$ is a complete lattice for all $p,q\in\mathcal{Q}_0$, such that

$$v\circ\Big(\bigvee_{i\in I}u_i\Big)=\bigvee_{i\in I}(v\circ u_i)\quad\text{and}\quad\Big(\bigvee_{i\in I}v_i\Big)\circ u=\bigvee_{i\in I}(v_i\circ u)$$

for all $u,u_i\in\mathcal{Q}(p,q)$, $v,v_i\in\mathcal{Q}(q,r)$ $(i\in I)$. The right adjoints induced by $\circ$ are denoted by $\swarrow$ and $\searrow$, respectively, which satisfy

$$v\circ u\leqslant w\iff v\leqslant w\swarrow u\iff u\leqslant v\searrow w$$

for all $\mathcal{Q}$-arrows $u\colon p\longrightarrow q$, $v\colon q\longrightarrow r$, $w\colon p\longrightarrow r$.

Maps in a quantaloid

Note that the right adjoint of a $\mathcal{Q}$-arrow $u\colon p \longrightarrow q$ in $\mathcal{Q}$, when it exists, is necessarily

$$u^* := u \searrow 1_q \colon q \longrightarrow p.$$

u is called a map in $\mathcal{Q}$ if $u \dashv u^*$. We denote by

$$\mathbf{Map}(\mathcal{Q})$$

the subcategory of $\mathcal{Q}$ whose objects are the same as $\mathcal{Q}$, and whose morphisms are maps in $\mathcal{Q}$.

H. Heymans. Sheaves on Quantales as Generalized Metric Spaces. *PhD thesis, Universiteit Antwerpen*, 2010.

Q-maps

Definition

A Q-map ζ from a set X to a set Y is a map

$$\zeta\colon X \rightarrow\!\!\!\!\!\!\mapsto Y$$

in the quantaloid $\mathcal{Q}\text{-}\mathbf{Rel}$.

Sets and Q-maps constitute a category

$$\mathcal{Q}\text{-}\mathbf{Map} := \mathbf{Map}(\mathcal{Q}\text{-}\mathbf{Rel}).$$

Q-maps

Let $\zeta\colon X \longrightarrow\!\!\!\!\!\mapsto Y$ be a Q-map.

- The value
$$\zeta(x, y)$$
is interpreted as the extent of y being the image of x under the map ζ.
- The value
$$\zeta^*(y, x) = (\zeta \searrow \mathrm{id}_Y)(y, x) = \bigwedge_{z \in Y} \zeta(x, z) \to \mathrm{id}_Y(y, z)$$
also represents the extent of y being the image of x under the map ζ, since the above expression may be understood as:
  - For each $z \in Y$, if z is the image of x under ζ, then z is equal to y.

Q-maps

Therefore, the adjunction $\zeta \dashv \zeta^*$ can be translated as follows:

- For every $x \in X$, there exists $y \in Y$ such that y is the image of x under ζ; because $\mathrm{id}_X \leqslant \zeta^* \circ \zeta$ means that
$$k \leqslant \bigvee_{y \in Y} \zeta^*(y,x) \,\&\, \zeta(x,y)$$
for all $x \in X$.
- If $y, z \in Y$ are both the images of x under ζ, then y is equal to z; because $\zeta \circ \zeta^* \leqslant \mathrm{id}_Y$ means that
$$\bigvee_{x \in X} \zeta(x,z) \,\&\, \zeta^*(y,x) \leqslant \mathrm{id}_Y(y,z)$$
for all $y, z \in Y$.

Symmetric Q-maps

A Q-map $\zeta\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ is symmetric if

$$\zeta^*(y,x) = \zeta(x,y)$$

for all $x \in X$, $y \in Y$.

In particular, every map $f\colon X \longrightarrow Y$ between sets induces a symmetric Q-map

$$f_\circ\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y, \quad f_\circ(x,y) = \begin{cases} k & \text{if } y = f(x), \\ \bot & \text{else,} \end{cases}$$

called the graph of f.

Questions

- Is every Q-map symmetric?
- Is every Q-map the graph of a map in $\mathbf{Set}$?

Lean and weakly lean quantales

Definition

Let Q be a non-trivial, commutative and unital quantale. We say that:

1. Q is lean, if

$$(p \vee q = k \text{ and } p \,\&\, q = \bot) \implies (p = k \text{ or } q = k)$$

and

$$p \,\&\, q = k \iff p = q = k$$

for all $p, q \in Q$;

D. Hofmann, G. J. Seal, and W. Tholen, editors. Monoidal Topology: A Categorical Approach to Order, Metric, and Topology. *Cambridge University Press*, 2014.

2. Q is weakly lean, if

$$\Big(\bigvee_{i\in I} p_i \mathbin{\&} q_i = k \ \text{ and } \ p_i \mathbin{\&} q_j = \bot \ (i \neq j)\Big)$$
$$\Longrightarrow \Big(k \leqslant \bigvee_{i\in I} (p_i \wedge q_i) \mathbin{\&} (p_i \wedge q_i)\Big)$$

for all $p_i, q_i \in \mathsf{Q}$ $(i \in I)$.

H. Heymans. Sheaves on Quantales as Generalized Metric Spaces. *PhD thesis, Universiteit Antwerpen*, 2010.

Lean and weakly lean quantales

Lemma

If Q *is lean, then* Q *is weakly lean.*

Lemma

If Q *is integral (i.e.,* $k = \top$*), then* Q *is weakly lean.*

Lean and weakly lean quantales

Example

On the three-chain

$$C_3 = \{\bot, k, \top\}$$

we have the non-integral lean quantale

$$(C_3, \&, k),$$

with

$$\top \& \top = \top$$

and the other multiplications being trivial.

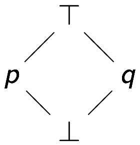
Lean and weakly lean quantales

Example

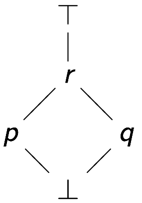
Every frame is an integral quantale, and thus weakly lean. Let

$$F_1 = \{\bot, p, q, \top\} \quad \text{and} \quad F_2 = \{\bot, p, q, r, \top\}$$

be the frames illustrated by the Hasse diagrams



and



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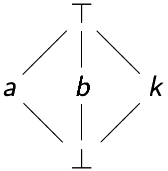
respectively. Then F_1 is not lean while F_2 is lean. Therefore, a weakly lean quantale need not be lean.

Lean and weakly lean quantales

Example

On the diamond lattice M_3 given by the Hasse diagram



we have the lean quantale $(M_3, \&, k)$, with

$$a \,\&\, a = b, \quad b \,\&\, b = a, \quad a \,\&\, b = k \quad \text{and} \quad a \,\&\, \top = b \,\&\, \top = \top.$$

Lean and weakly lean quantales

Example

For each continuous t-norm $*$ on the unital interval $[0,1]$, the quantale

$$([0,1], *, 1)$$

is lean.

Lean and weakly lean quantales

Example

Let $[-\infty, \infty]$ be the extended real line equipped with the order "$\geqslant$". Then the quantale

$$([-\infty, \infty], +, 0)$$

is not weakly lean, while the Lawvere quantale

$$([0, \infty], +, 0)$$

is lean.

Lean and weakly lean quantales

Example

Consider the free quantale

$$(\mathsf{P}M, \&, \{k\})$$

induced by a commutative monoid $(M, \&, k)$:

- $(\mathsf{P}M, \&, \{k\})$ is lean if, and only if, k is the only element of M with an inverse.
- $(\mathsf{P}M, \&, \{k\})$ is weakly lean if, and only if, there exist no $m, m' \in M$ such that $m \neq m'$ and $m \& m' = k$.

In particular, the free quantale induced by the cyclic group

$$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

is non-integral, weakly lean, but not lean.

The symmetry of Q-maps

Theorem

Every Q-map is symmetric if, and only if, Q is weakly lean.

H. Heymans. Sheaves on Quantales as Generalized Metric Spaces. *PhD thesis, Universiteit Antwerpen*, 2010.

Q-$\mathbf{Map}$ and $\mathbf{Set}$

Theorem

Every Q*-map is the graph of a map in* $\mathbf{Set}$ *if, and only if,* Q *is lean. In this case,* Q-$\mathbf{Map}$ *and* $\mathbf{Set}$ *are isomorphic categories.*

D. Hofmann, G. J. Seal, and W. Tholen, editors. Monoidal Topology: A Categorical Approach to Order, Metric, and Topology. *Cambridge University Press*, 2014.

Towards partial Q-maps

Let $\mathcal{C}$ be a category with finite coproducts. For every $\mathcal{C}$-object A, the forgetful functor

$$\mathsf{U}\colon A/\mathcal{C} \longrightarrow \mathcal{C}, \quad (A \to C) \mapsto C$$

admits a left adjoint

$$\mathsf{F}\colon \mathcal{C} \longrightarrow A/\mathcal{C}, \quad C \mapsto (A \to A \amalg C).$$

The Eilenberg-Moore category of the induced monad is isomorphic to $A/\mathcal{C}$, and thus $A/\mathcal{C}$ is monadic over $\mathcal{C}$.

S. Mac Lane. Categories for the Working Mathematician. *Springer*, 1998.

Partial Q-maps

It is straightforward to check that $\mathcal{Q}\text{-}\mathbf{Map}$ has all coproducts. Let

$$X_+ := X \amalg \{\star\}.$$

Definition

A partial Q-map ζ from a set X to a set Y is a Q-map

$$\zeta\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y_+.$$

The maybe monad on Q-$\mathbf{Map}$

The adjunction

$$\text{Q-}\mathbf{Map} \underset{\mathsf{U}}{\overset{\mathsf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \{\star\}/\text{Q-}\mathbf{Map}$$

induces the maybe monad

$$(\mathsf{T}, \mathsf{m}, \iota)$$

on Q-$\mathbf{Map}$, whose Eilenberg-Moore category Q-$\mathbf{Map}^{\mathsf{T}}$ is isomorphic to $\{\star\}/$Q-$\mathbf{Map}$.

The category of partial Q-maps

Note that

- objects of the Kleisli category $\mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$ are sets, and
- a morphism from X to Y in $\mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$ is exactly a Q-map $X \longrightarrow\!\!\!\!\!\!\mapsto Y_{+}$; that is, a partial Q-map from X to Y.

Thus, we denote by

$$\mathsf{Q}\text{-}\mathbf{ParMap} := \mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}}$$

the category of sets and partial Q-maps.

Q-$\mathbf{ParMap}$ and $\mathbf{Set}^{\partial}$

Theorem

Every partial Q*-map is the graph of a partial map in* $\mathbf{Set}$ *if, and only if,* Q *is lean. In this case,* Q-$\mathbf{ParMap}$ *and* $\mathbf{Set}^{\partial}$ *are isomorphic categories.*

The monadicity of Q-$\mathbf{ParMap}$ over Q-$\mathbf{Map}$

Theorem

Assuming the axiom of choice, every T*-algebra is free. Therefore, the Kleisli category*

$$\mathsf{Q}\text{-}\mathbf{Map}_{\mathsf{T}} = \mathsf{Q}\text{-}\mathbf{ParMap}$$

and the Eilenberg–Moore category

$$\mathsf{Q}\text{-}\mathbf{Map}^{\mathsf{T}} \cong \{\star\}/\mathsf{Q}\text{-}\mathbf{Map}$$

of the maybe monad $(\mathsf{T}, \mathsf{m}, \iota)$ *on* $\mathsf{Q}\text{-}\mathbf{Map}$ *are equivalent.*

Corollary

Assuming the axiom of choice, $\mathsf{Q}\text{-}\mathbf{ParMap}$ *is monadic over* $\mathsf{Q}\text{-}\mathbf{Map}$*.*

Thank you!