Resolvable and irresolvable spaces

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The beginning

- Hewitt, Ph.D Thesis, Harvard, 1942
- **Def.** A space X is **resolvable** iff X contains two disjoint dense subsets.
- **Def.** A space is **irresolvable** iff it is not resolvable.
- **What makes a space (ir)resolvable?**
- **Fact.** If X has an isolated point, then X is irresolvable
- Revised question: **What makes a crowded space (ir)resolvable?**

The beginning

- **Are there irresolvable crowded spaces?**
- the structure of refinements of a topology
- Assume that $\langle X, \tau \rangle$ is a crowded 0-dimensional space
- Let $\tau' \supset \tau$ be a **maximal 0-dimensional crowded** topology on X.
- **Claim:** $\langle X, \tau' \rangle$ is irresolvable.
- Proof. Assume on the contrary that X has a partition $\{D_0, D_1\}$ into dense sets
- You can refine the topology τ' by declaring that D_0 and D_1 are open!
- $\tau^* = \langle \tau' \cup \{D_0, D_1\} \rangle_{gen}$ is 0-dimensional crowded
- Contradiction: τ' was not maximal.

- **Fact**: If X is a space, then the closed subspace
$$Res(X) = \overline{\bigcup\{Y \subset X : Y \text{ is resolvable}\}}$$
is resolvable.
- **Fact.** If X is irresolvable, then $X \setminus Res(X)$ is a non-empty open, hereditarily irresolvable subspace.
- **Def.** A space is hereditarily irresolvable (HI) iff every crowded subspace is irresolvable
- **Fact.** If X is a space and every non-empty open subset of X contains a resolvable subspace, then X is resolvable.
- **Fact.** Assume that $\mathbb{K}$ is a family of regular spaces which is closed for regular-closed subsets. If every $X \in \mathbb{K}$ contains a resolvable subspace, then every $X \in \mathbb{K}$ is resolvable.

- **Def.** A topological space is **κ-resolvable** iff X contains κ disjoint dense subsets.
- **Fact**: If X is a space and κ is a cardinal, then the closed subspace

$$Res_\kappa(X) = \overline{\bigcup\{Y \subset X : Y \text{ is } \kappa\text{-resolvable}\}}$$

is κ-resolvable.
- If D is **dense** and U is a non-empty **open** set, then $U \cap D \neq \emptyset$.
- **if X is κ-resolvable then** $\kappa \leq \min\{|U| : U \in \tau_X^+\} = \Delta(X)$.
- **Def.** $\Delta(X)$ is the dispersion character of X.
- **Def.** A space X is maximally resolvable iff it is $\Delta(X)$-resolvable.

The beginning

A space X is **maximally resolvable** iff it is $\Delta(X)$-resolvable.

Thm. A topological space X is **maximally resolvable** provided it is

- metric, or
- ordered, or
- compact.

What about

- **monotonically normal** spaces?
- **Lindelöf** spaces?
- **countably compact** spaces?
- **pseudocompact** spaces?

The expectation is that *nice* spaces should be maximally resolvable.

Neat spaces

- **Def.** A topological space X is **neat** iff $|X| = \Delta(X)$.
- Every regular space contains a regular-closed neat subspace.
- Assume that $\mathbb{K}$ is a class of regular spaces which is closed for regular-closed subspaces.
  - If every neat $X \in \mathbb{K}$ is κ-resolvable, then every $X \in \mathbb{K}$ is κ-resolvable.
  - If every neat $X \in \mathbb{K}$ is maximally resolvable, then every $X \in \mathbb{K}$ is maximally resolvable.
- The class of compact (or countably compact, Lindelöf, monotonically normal, pseudocompact) spaces is closed for regular-closed subspaces.
- **it is enough to investigate the resolvability of neat spaces.**

X is **neat** iff $|X| = \Delta(X)$.

- **Fact.** If X is neat and $\pi(X) \leq |X|$, then X is maximally resolvable.
- **Proof**: Write $\kappa = \pi(X) \leq |X| = \Delta(X) = \lambda$
- Let $\{B_\eta : \eta < \kappa\}$ be a π-base
- By transfinite recursion choose distinct points $\{d_{\xi,\eta} : \eta < \kappa, \xi < \lambda\}$ such that $d_{\xi,\eta} \in B_\eta$.
- Put $D_\xi = \{d_{\xi,\eta} : \eta < \kappa\}$ for $\xi < \lambda$.
- $\{D_\xi : \xi < \lambda\}$ is a family of pairwise disjoint dense sets. QED
- **Fact.** Neat, compact crowded spaces are maximally resolvable.
- **Thm.** Compact spaces are maximally resolvable.

- **Fact.** If X is neat and $t(X) < |X|$, then X is maximally resolvable.
- **Proof**. Write $\kappa = |X| = \Delta(X)$.
- If $A \in [X]^{<\kappa}$, then there is $B \subset X \setminus A$ with $A \subset \overline{B}$ and $|B| \leq |A| \cdot t(X)$.
- Write $X = \{x_\alpha : \alpha < \kappa\}$.
- By transfinite recursion on α, construct disjoint sets $\{D^\alpha_\xi : \xi < \alpha < \kappa\}$ such that $\{x_i : i < \alpha\} \subset \overline{D^\alpha_\xi}$ and $|D^\alpha_\xi| \leq |\alpha| \cdot t(X)$.
- Let $D_\xi = \bigcup\{D^\alpha_\xi : \xi < \alpha < \kappa\}$.
- $\{D_\xi : \xi < \kappa\}$ is a family of pairwise disjoint dense sets.
- **Corollary:** (Neat) metric spaces with $\Delta(X) > \omega$ are maximally resolvable.
- **Thm.** Metric spaces are maximally resolvable.

How to prove resolvability
define dense subsets by properties

- **Fact.** Every crowded, countably compact, regular space is resolvable.
- **Proof:** Let X be a crowded, countably regular compact space.
- **Def.** A subset $D \subset X$ is **strongly discrete (SD)** if there is a neighborhood assignment $U : D \to \tau_X$ such that the sets $\{U(d) : d \in D\}$ are pairwise disjoint.
- **Def.** A point $x \in X$ is an **SD-point** iff there is an SD set $D \subset X$ such that $x \in D'$.
- Let $A = \{x \in X : x$ **is an SD-point**$\}$.
- A is dense in X.
- If $B = X \setminus A$ is dense, then we are done.
- Assume that B is not dense. So A contains a non-empty open set $U \in \tau_X^+$.
- Construct $T = \{x_s : s \in \omega^{<\omega}\} \subset U$ by induction on $|s|$ such that for each $s \in \omega^{<\omega}$
    - $\{x_{s^\frown n} : n < \omega\}$ is SD, and $x_s \in \{x_{s^\frown n} : n < \omega\}'$.

Monotonically normal spaces

- **Def.** Given a space X, define the family of marked open sets as follows:
  $$\mathcal{M}(X) = \{\langle x, U\rangle \in X \times \tau(X) : x \in U\}$$
- **Def.** A T_1 space X is monotonically normal (MN) if X admits a monotone normality operator i.e. there is function $H : \mathcal{M}(X) \to \tau(X)$ such that
  - $x \in \mathrm{H}(x, U) \subset U$,
  - if $x \notin V$ and $y \notin U$ then $\mathrm{H}(x, U) \cap \mathrm{H}(y, V) = \emptyset$.
- Metric and linearly ordered spaces are MN
- Are the monotonically normal spaces maximally resolvable?
- **Thm.** A dense-in-itself monotonically normal space is ω-resolvable.
- **Fact.** In a crowded monotonically normal space every point is an SD-point.

- **Problem** (Ceder, Pearson 1967) Does ω**-resolvable** imply **maximally resolvable**?

A simple construction of MN spaces

X dense-in-itself, monotonically normal $\stackrel{?}{\Longrightarrow}$ X maximally resolvable?

- If κ is an infinite cardinal, and F is an **ultrafilter** on κ define the space T_F as follows
- The underlying set is the everywhere κ-branching tree of height ω: $\kappa^{<\omega}$.
- $U \subset \kappa^{<\omega}$ is **open** iff for each $t \in U$ the set $\{\alpha < \kappa : t^\frown\alpha \in U\} \in F$.
- T_F is monotonically normal: $\mathrm{H}(t, V) = \{u \in V : [t, u] \subset V\}$
- T_F is ω-resolvable: if $I \subset \omega$ is infinite, then $\{s \in \kappa^{<\omega} : |s| \in I\}$ is dense in T_F.

Natural conjecture : **If U is a uniform ultrafilter on ω_1 then T_F is not ω_1-resolvable**

Consistent counterexamples

Ceder - Pearson: Does ω**-resolvable** imply **maximally resolvable**?

X dense-in-itself, monotonically normal $\overset{?}{\Longrightarrow}$ X maximally resolvable?

Conjecture: If F is a uniform ultrafilter on ω_1 then T_F is **not** ω_1**-resolvable**

- Thm (Juhász-S-Szentmiklóssy) If $\mathcal{F}$ is a uniform ultrafilter on some $\kappa < \aleph_\omega$, then $T_\mathcal{F}$ is maximally resolvable.
- Thm (Juhász-S-Szentmiklóssy) Assume that $\kappa = cf(\kappa) \geq \lambda$. Then the following are equivalent.
  - Every MN space with $|X| = \Delta(X) = \kappa$ is λ-resolvable.
  - For every uniform ultrafilter F on κ, the space T_F is λ-resolvable.
- Corollary. Every MN space with $|X| < \aleph_\omega$ is maximally resolvable.
- Thm: (Juhász-Magidor) The following are equiconsistent:
  - There is a MN space that is **not maximally resolvable**.
  - There is a MN space X with $|X| = \Delta(X) = \aleph_\omega$ that is **not** ω_1-resolvable.
  - There is a **measurable cardinal**.

Resolvability of Lindelöf spaces.

- Since there are countable irresolvable spaces, and countable spaces are clearly Lindelöf, the following question of Malyhin is the natural one:
- Is it true that every Lindelöf space with $\Delta(X) > \omega$ is resolvable?
- Pavlov: Any Lindelöf space X with $\Delta(X) > \omega_1$ is ω-resolvable.
- Filatova: Any regular Lindelöf space X with $\Delta(X) = \omega_1$ is 2-resolvable.
- Juhász,S,Szentmiklóssy: *a Lindelöf space with $|X| = \Delta(X) = \omega_1$ is maximally resolvable.*
- Open: Is it true that every regular Lindelöf space of uncountable dispersion character is maximally resolvable?

Resolvability of countably compact spaces

- E. G. Pytkeev proved that **crowded countably compact regular spaces are ω_1-resolvable**
- different proof working for crowded countably compact π-regular spaces.
- However, no one of these proofs can reach either 2^ω-resolvability or maximal resolvability.
- **Is it true that the countably compact, crowded regular spaces are maximally resolvable?**

Resolvability of pseudocompact (feebly compact) spaces

- a tougher problem: weaken countably compact to pseudocompact.
- **Thm.** A crowded pseudocompact space X is 2^ω-resolvable provided
  (1) $c(X) = \omega$ (J. van Mill, 2016)
  (2) $c(X) \le 2^\omega$ (A. H. Ortiz-Castillo and Y. F. Tomita (2018))
  (3) every complete, crowded metric space M with density $\le c(X)$ has a "**Bernstein coloring**", i.e. a coloring $f : M \to 2^\omega$ such that $f''C = 2^\omega$ provided $C \subset M$ is a topological copy of the Cantor set. (Juhász,S,Szentmiklóssy)
- Hajnal, Juhász and Shelah: the failure of (3) is a large cardinal assumption.
- Hence, **it is consistent that every crowded pseudocompact space is 2^ω-resolvable.**
- **Is it true that every pseudocompact, regular spaces X is resolvable (2^ω-resolvable)?**
- **Is it true (or just consistent) that every pseudocompact, regular spaces X is maximally resolvable?**

Between pseudocompactness and countably compactness

What can we say about resolvability of spaces with property P, where P is a property strictly stronger than pseudocompactness, but strictly weaker than countable compactness?

A zoo of spaces.

- X is countably compact
- X contains a dense subset D such that every $A \in [D]^{\omega}$ has an accumulation point
  (D is relatively countably compact in X)
- X contains dense subsets $\{D_n : n < \omega\}$ such that if $a_n \in D_n$ for $n < \omega$, then $A = \{a_n : n < \omega\}$ has an accumulation point.
- Player II has a winning strategy in the following game:
  in the n^{th} turn of the game Player I chooses $U_n \in \tau_X^+$, then Player II selects $a_n \in U_n$.
  Player II wins iff $A = \{a_n : n < \omega\}$ has an accumulation point.
- if $\{U_n : n < \omega\} \subset \tau_X^+$, then there are points $a_n \in U_n$ for $n < \omega$ such that $A = \{a_n : n < \omega\}$ has an accumulation point.
  (X is sequentially pseudocompact)

Resolvability of product

- Any product of infinitely many non-singleton spaces is 2^ω-resolvable
- What about the product of two crowded spaces?
- **Fact.** If X and Y are neat, $|X| = \Delta(X) = |Y| = \Delta(Y)$, then $X \times Y$ is resolvable:
- Proof: Write $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$.
- Let $A = \{\langle x_\alpha, y_\beta \rangle : \alpha < \beta < \kappa\}$ and $B = \{\langle x_\alpha, y_\beta \rangle : \beta \leq \alpha < \kappa\}$.
- A and B are dense subsets in $X \times Y$
- **Problem: Assume that $|X| = \Delta(X) = |Y| = \Delta(Y)$. Is it true that $X \times Y$ is 3-resolvable?**

Resolvability of product

- **Fact.** If X is κ-resolvable, then $X \times Y$ is κ-resolvable.
- **Thm.** If X is κ-resolvable, and $|Y| = \Delta(Y) = \kappa^+$, then $X \times Y$ is κ^+-resolvable.
- Problem(Ceder and Pearson):**Is the product of a maximally resolvable space with any other space maximally resolvable?**
- Eckertson gave a consistent counterexample modulo the existence of a measurable cardinal.
- We also gave two counterexamples with some additional properties. Both constructions used the existence of a measurable cardinal.
- **Do we really need large cardinals to construct counteraxamples for Ceder-Pearson problem?**
- The intuition is that the answer should be no.
- Answering Malyhin's question we proved that the following are equiconsistent:
  - there are two crowded 0-dimensional T_2 -spaces whose product is irresolvable.
  - the existence of a measurable cardinal.

Stepping up in resolvability

- **Thm (Ceder and Pearson)**: there is an n-resolvable, but not $n+1$ resolvable space for each $1 \leq n < \omega$.
- **Hint**. Construct a space X_n which is the disjoint union of n dense, HI subsets.
- Then X_n is not n+1-resolvable.
- **Thm: (Illanes)** If a space X is n-resolvable for each $n \in \omega$, then X is ω-resolvable as well.
- Problem (Ceder and Pearson): Is there a space X with $\Delta(X) > \omega$ such that X is ω-resolvable, but not ω_1-resolvable? (maximally resolvable)
- Assume that a space X is the union of countable many dense, HI subspaces. Is it true that X is not ω_1-resolvable?
- consistent counterexamples (El'kin, Malykhin, Eckertson, and Hu)
- Juhász-S-Szentmiklóssy: $\mathcal{D}$**-forced spaces.**

Stepping up in resolvability

- **Def.** If X is a topological space, $\mathcal{D}$ is a family of subsets of X, we define the notion of **$\mathcal{D}$-mosaic** as follows: if $\mathcal{U}$ is a maximal cellular family of open sets, and $D_U \in \mathcal{D}$ for each $U \in \mathcal{U}$, then the set

$$M(\mathcal{D},\mathcal{U}) = \bigcup\{U \cap D_U : U \in \mathcal{U}\} \text{ is a } \mathcal{D}\textbf{-mosaic}.$$

- **Fact.** if $\mathcal{D}$ is a family of dense sets, then $M(\mathcal{D},\mathcal{U})$ is dense.
- **Thm.** Assume that
  - $X = \langle \kappa, \tau \rangle$ is a homeomorphic to a dense subset of $D(2)^{2^\kappa}$, and
  - $\mathcal{D}$ is a family of τ-dense subsets of X.

  Then we can modify the topology τ to obtain a topology τ' such that
  - $X' = \langle \kappa, \tau' \rangle$ is homeomorphic to a dense subset $D(2)^{2^\kappa}$, X' is nodec,

  and
  - a set $A \subset \kappa$ is τ'-dense iff it contains a $\mathcal{D}$-mosaic.
- **Corr.** For each infinite cardinal λ there is 0-dimensional T_2 space X which is is λ-resolvable, but not λ^+-resolvable.
- **Proof** $X \subset D(2)^{2^\lambda}$ dense, $|X| = \lambda$, $X = \bigcup\{D_\alpha : \alpha < \lambda\} \subset Dense(X)$, pairwise disjoint.

Stepping up in resolvability

$\forall\lambda \geq \omega$ there is 0-dimensional T_2 space X which is is λ-resolvable, but not λ^+-resolvable.

- Natural question: Assume that λ is a limit cardinal and a space X is μ-resolvable for each $\mu < \lambda$. Should X be λ-resolvable as well?
- Illanes: YES provided $\lambda = \omega$.
- Bashkara Rao: YES provided $cf(\lambda) = \omega$
- Juhász,S,Szentmiklóssy: NO provided λ is regular (that is, inaccessible).
- Assume that λ is a singular cardinal with cf(λ) > ω and X is a topological space that is μ-resolvable for all μ < λ. Is it true then that X is also λ-resolvable?

link

Resolvability in c.c.c generic extensions

Adrienne Stanley, S

- Every crowded space X is ω-resolvable in $V^{Fn(|X|,2)}$.
- What we can say about λ-resolvability for $\lambda > \omega$?
- A topological space is *monotonically ω_1-resolvable* if there is a function $f : X \to \omega_1$ such that for each $\alpha < \omega_1$:

$$\{x \in X : f(x) \geq \alpha\} \subset^{dense} X.$$

- **Thm.** (Adrienne Stanley, S) TFAE:
  - X is ω_1-resolvable in some c.c.c. generic extension;
  - X is monotonically ω_1-resolvable;
  - X is ω_1-resolvable in the Cohen-generic extension $V^{Fn(\omega_1,2)}$.
- **Thm.** (St, S) If X is c.c.c., and $\omega_1 \leq \Delta(X) \leq |X| < \aleph_\omega$, then X is monotonically ω_1-resolvable.
- **Thm.** (St, S) It is consistent, modulo the existence of a measurable cardinal, that there is a space Y with $|Y| = \Delta(Y) = \aleph_\omega$ which is not monotonically ω_1-resolvable.
- Natural question: is it true that crowded spaces from the ground model

- **Thm.** (A. H. Stone): Every partially ordered set $(P, \leq)$ without maximal elements can be partitioned into two cofinal subsets.
- **Corr.** Every π-base of a crowded space can be decomposed into two π-bases.
- **Def.** A space X is base resolvable if every base of X can be decomposed into two bases.
- **Thm.** Every crowded metric space is base resolvable.
- **Thm.** (D. Soukup) Crowded Lindelöf spaces are base resolvable
- **Thm.** (S) It is consistent that there is a first countable, 0-dimensional, T_2 space which is not base-resolvable.
- **ZFC example?**
- What about linearly ordered spaces? (Hereditarily) separable spaces? Paracompact spaces?

Thank you.