Game-theoretic results in regards to cardinal functions

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Why topological games?

Theorem (Arhangel'skii)

Let X be a Hausdorff space. Then, $|X| \leq 2^{\chi(X)L(X)}$. In particular, if X is a Hausdorff, Lindelöf, first-countable space, then $|X| \leq 2^{\aleph_0}$.

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Is $|X| \leq 2^{\aleph_0}$ for all Lindelöf spaces with point G_δ? What about for all T_1 first-countable spaces?

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There is a model in which CH holds, $2^{\aleph_1}$ is arbitrarily large and where there is a T_2, zero-dimensional Lindelöf space with points G_δ and cardinality $2^{\aleph_1}$.

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Theorem (Scheepers, Tall)

If X be a space with points G_δ such that $\mathrm{II} \uparrow G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$, then $|X| \leq 2^{\aleph_0}$.

Why the cellularity game?

Definition

The cellularity game of length κ, $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$ is the game between players I and II such that, at each inning $\alpha < \kappa$, I plays some $\mathcal{U}_\alpha \in \mathcal{O}_D$, II picks some $U_\alpha \in \mathcal{U}_\alpha$ and II wins if $\{U_\alpha : \alpha < \kappa\} \in \mathcal{O}_D$.

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Definition

Given a space X, we define:
$c(X) := \sup\{|\mathcal{C}| : \mathcal{C}$ is a cellular family of $X\} \cdot \omega$;
$\hat{c}(X) := \min\{\kappa : X$ has no cellular family of size $\kappa\}$.

Lemma

For any space X, $II \uparrow G_1^{\hat{c}(X)}(\mathcal{O}_D, \mathcal{O}_D)$.

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Remark

$II \uparrow G_1^\omega(\mathcal{O}_D, \mathcal{O}_D) \longrightarrow c(X) = \omega \longrightarrow II \uparrow G_1^{\omega_1}(\mathcal{O}_D, \mathcal{O}_D) \longrightarrow c(X) \leq \omega_1$

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Let X be a regular space such that $\psi(X) = \omega$, $\pi\chi(X) \leq 2^\omega$ and $II \uparrow G_1^{\omega_1}(\mathcal{O}_D, \mathcal{O}_D)$. Is it necessarily true that $|X| \leq 2^\omega$?

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Proposition

($2^{\omega_1} = 2^\omega$) Let X be a regular space such that $\psi(X) = \omega$, $\pi\chi(X) \leq 2^\omega$ and $II \uparrow G_1^{\omega_1}(\mathcal{O}_D, \mathcal{O}_D)$. Then, $|X| \leq 2^{\omega_1} = 2^\omega$.

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Lemma

If $2^{<\kappa} = \kappa$*, then there is a almost disjoint family* $\mathcal{A} \subset [\kappa]^\kappa$ *such that* $|\mathcal{A}| = 2^\kappa$*.*

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For each $A \in \mathcal{A}$, we construct a uniform ultrafilter $\mathcal{U}_A \subset [\kappa]^\kappa$ such that $A \in \mathcal{U}_A$ and define the space $X := D(\kappa) \cup \{\mathcal{U}_A : A \in \mathcal{A}\}$ with the topology generated by $\{[S] : S \subset \kappa\}$, where $[S] = S \cup \{\mathcal{U}_A : A \in \mathcal{A} \wedge S \in \mathcal{U}_A\}$. Thus X is a subspace of $\beta(D(\kappa))$.

Theorem

(GCH) If κ *is a successor cardinal, then there is a Tychonoff space* X *such that* $\psi(X) < \kappa$*,* $\pi\chi(X) \leq \kappa = 2^{<\kappa}$ *and* $II \uparrow G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$*, but* $|X| = 2^\kappa > \kappa = 2^{<\kappa}$*.*

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Let X be a regular space such that $\psi(X) = \omega$, $\pi\chi(X) \leq 2^\omega$ and $II \uparrow G_1^{\omega_1}(\mathcal{O}_D, \mathcal{O}_D)$. Is it necessarily true that $|X| \leq 2^\omega$?

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(CH) There is a Tychonoff space X such that $\psi(X) = \omega$, $\pi\chi(X) \leq \omega_1 = 2^\omega$ and $II \uparrow G_1^{\omega_1}(\mathcal{O}_D, \mathcal{O}_D)$, but $|X| = 2^{\omega_1} > \omega_1 = 2^\omega$.

Some classic results

Theorem (Hajnal, Juhász)

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Definition

- $L(X)$ is the least infinite cardinal κ such that given an open cover of X, there is a subcover of cardinality $\leq \kappa$
- $c(X) = \sup\{|\mathcal{C}| : \mathcal{C}$ is a cellular family of $X\}$
- $wL(X)$ is the least infinite cardinal κ such that given an open cover $\mathcal{U}$ of X, there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $\overline{\bigcup \mathcal{V}} = X$
- $\chi(X)$ is the least infinite κ such that every point of X has a local base of cardinality $\leq \kappa$

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Question (Bell, Ginsburg, Woods)

Is it true for every regular space X that $|X| \leq 2^{\chi(X)wL(X)}$? Is $|X| \leq 2^{t(X)\psi(X)wL(X)}$ for every normal space X?

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Theorem (Aurichi, Bella, Spadaro)

Let X be a Urysohn space such that $\chi(X) < \kappa$, $\mathrm{II} \uparrow G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$. Then, $|X| \leq 2^{<\kappa}$.

A few improvements

Theorem (Gotchev, Tkachuk, Tkachenko)

Let X be a Hausdorff space. Then $|X| \leq \pi w(X)^{ot(X).\psi_c(X)}$.

Theorem (Bella, —, Spadaro)

Let X be a regular space such that $ot(X) \cdot \psi(X) < \kappa$, $\chi(X) \leq 2^{<\kappa}$ and $II \uparrow G_1^{\kappa}(\mathcal{O}, \mathcal{O}_D)$. Then, $|X| \leq 2^{<\kappa}$.

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Theorem (Spadaro)

Let X be a regular space with a dense set of points of π-character $\leq 2^{<\kappa}$ where player II has a winning strategy in $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$. Then $\pi w(X) \leq 2^{<\kappa}$.

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Theorem (Bella, —, Spadaro)

Let X be a regular space such that $ot(X) \cdot \psi(X) < \kappa$, $\chi(X) \leq 2^{<\kappa}$ and $II \uparrow G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$. Then, $|X| \leq 2^{<\kappa}$.

Corollary (Spadaro)

Let X be a regular space with a dense set of points of π-character $\leq 2^{<\kappa}$ such that $ot(X) \cdot \psi(X) < \kappa$, $II \uparrow G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$. Then $|X| \leq 2^{<\kappa}$.

Theorem

There is a regular space which has a dense set of points of countable π-character such that $t(X) \cdot \psi(X) = \omega$ and $II \uparrow G_1^\omega(O, \mathcal{O}_D)$, but its cardinality is arbitrarily large.

Thanks!

Thank you for your attention!

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