On completeness-type properties of hyperspace topologies

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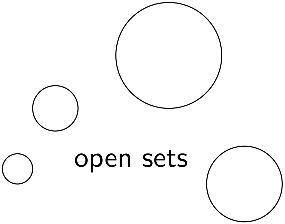

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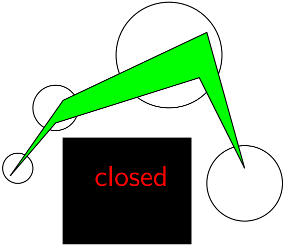

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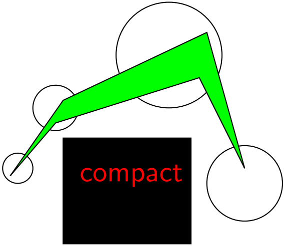

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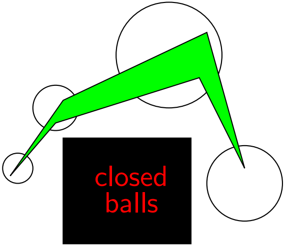

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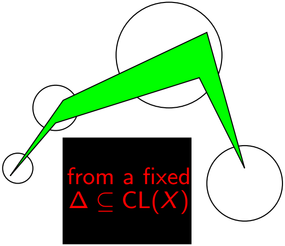

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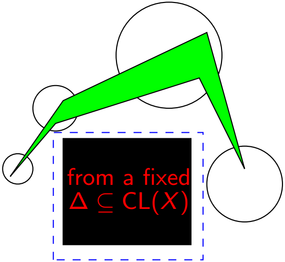

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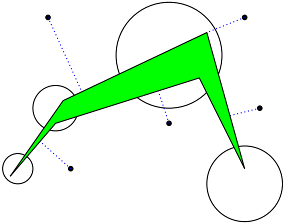

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$[S, A_0](\varepsilon) = \{A_1 \in CL(X) : S \cap A_0 \subseteq S_d(A_1, \varepsilon) \text{ and } S \cap A_1 \subseteq S_d(A_0, \varepsilon)\}.$

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- $(CL(X), \tau_{\mathcal{S}})$ completely metrizable $\Leftrightarrow$ (X, d) completely metrizable $+$ completion reminder of X separable $+$ conditions on $\mathcal{S}$ that make $\tau_{\mathcal{S}}$ metrizable **(Zs.)**

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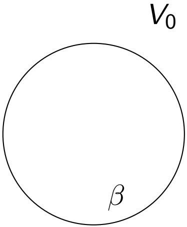

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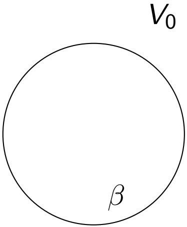

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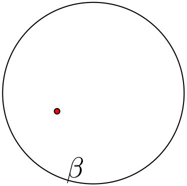

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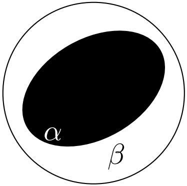

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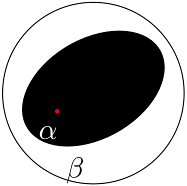

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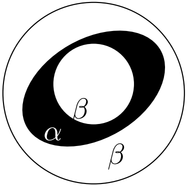

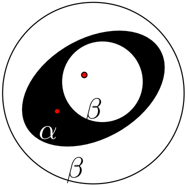

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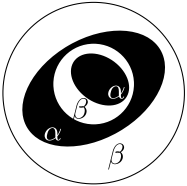

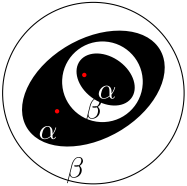

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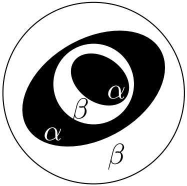

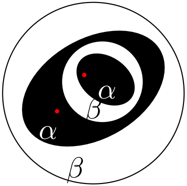

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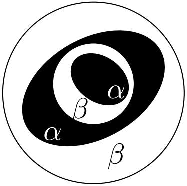

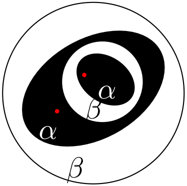

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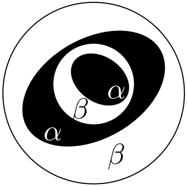

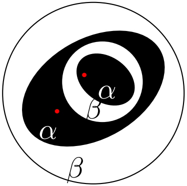

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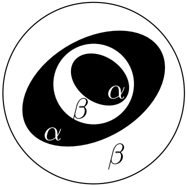

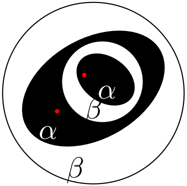

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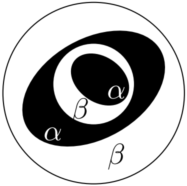

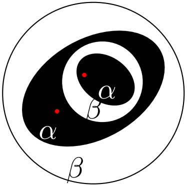

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- β – favorable $Ch(X) \Leftrightarrow$ X not hereditarily Baire
- α – favorable $Ch(X) \Leftrightarrow$ X completely metrizable

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- $(CL(X), \tau_{H_d})$ is α-favorable $\Leftarrow$ (X, d) separable + α-favorable
  - $\exists (X, d)$ (Bernstein set) separable+non-α-favorable with $(CL(X), \tau_{H_d})$ α-favorable

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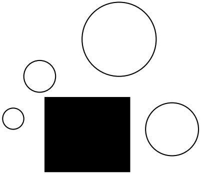
2. (X, d) Baire

Sketch of the idea behind the proofs

Hit-and-miss topology on $CL(X)$



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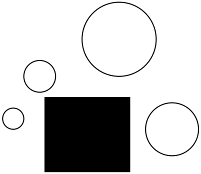

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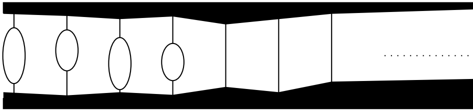

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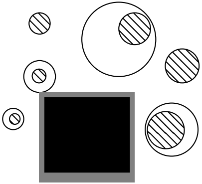

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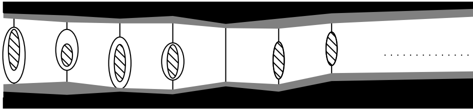

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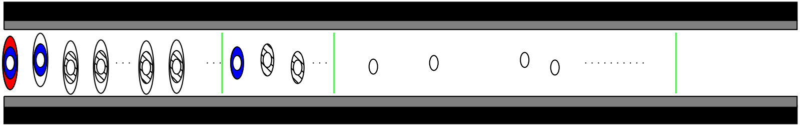

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