Vietoris endofunctor for closed relations and its de Vries dual

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Based on a homonymous paper with G. Bezhanishvili and L. Carai
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Vietoris Modalities: $\Box$, $\Diamond$

Main message: Vietoris has a nice dual.*

*Even for closed relations.

Stone duality

Stone spaces
(= comp. Hausd. + clopens
separate points)

Boolean algebras

$$X \mapsto \mathrm{Clopens}(X)$$

$$\begin{tikzcd}
\text{Stone} \arrow[r, bend left] \arrow[r, phantom, "\mathrm{op}"] \arrow[loop below, "\mathbb{V}"] & \text{BA} \arrow[l, bend left] \arrow[loop below, "\mathbb{K}"]
\end{tikzcd}$$

Vietoris provides Kripke semantics for modal logic.

Algebras for $\mathbb{K}$ = modal algebras.

Coalgebras for $\mathbb{V}$ = descriptive frames.

Stone spaces		Boolean algebras
Vietoris hyperspace	$\leftrightsquigarrow$	Free first layer of modality

$$X \rightsquigarrow \mathbb{V}(X)$$

$$A \rightsquigarrow \mathbb{K}(A) = \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\text{modal algebras axioms}}$$

The clopens of the Vietoris hyperspace $\mathbb{V}(X)$ of a Stone space X are all Boolean combinations of

$$\Diamond U := \{K \in \mathbb{V}(X) \mid K \cap U \neq \varnothing\}, \qquad U \text{ clopen of } X$$

$$\Box U := \{K \in \mathbb{V}(X) \mid K \subseteq U\} \qquad U \text{ clopen of } X.$$

Stone spaces		Boolean algebras
Vietoris hyperspace	$\leftrightsquigarrow$	Free first layer of modality

$$X \rightsquigarrow \mathbb{V}(X) \qquad\qquad A \rightsquigarrow \mathbb{K}(A) = \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\text{modal algebras axioms}}$$

$$f\colon X \to Y \text{ cont. funct.} \rightsquigarrow f[-]\colon \mathbb{V}(X) \to \mathbb{V}(Y) \qquad\qquad g\colon A \to B \text{ Bool. hom.} \rightsquigarrow \mathbb{K}(g)\colon \mathbb{K}(A) \to \mathbb{K}(B)$$

$$\mathbb{K}(g)\colon \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\text{modal algebras axioms}} \longrightarrow \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_b, \Diamond_b \mid b \in B\})}{\text{modal algebras axioms}}$$

$$[\Diamond_a] \longmapsto [\Diamond_{g(a)}]$$

$$[\Box_a] \longmapsto [\Box_{g(a)}]$$

But...

Stone spaces		Boolean algebras
Vietoris hyperspace	$\rightsquigarrow$	Free first layer of modality

$$\begin{array}{ccc}
X & \qquad & A \\
\rightsquigarrow & & \rightsquigarrow \\
\mathbb{V}(X) & & \mathbb{K}(A) = \dfrac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\text{modal algebras axioms}} \\[2em]
f\colon X \to Y \text{ cont. funct.} & & g\colon A \to B \text{ Bool. hom.} \\
\rightsquigarrow & & \rightsquigarrow \\
f[-]\colon \mathbb{V}(X) \to \mathbb{V}(Y) & & \mathbb{K}(g)\colon \mathbb{K}(A) \to \mathbb{K}(B)
\end{array}$$

$$\mathbb{K}(g)\colon \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\text{modal algebras axioms}} \longrightarrow \frac{\mathrm{Free}_{\mathrm{BA}}(\{\Box_b, \Diamond_b \mid b \in B\})}{\text{modal algebras axioms}}$$

$$[\Diamond_a] \longmapsto [\Diamond_{g(a)}]$$

$$[\Box_a] \longmapsto [\Box_{g(a)}]$$

$$
\begin{tikzcd}
\mathsf{KHaus} \arrow[loop below, "\mathbb{V}"] \arrow[r, bend left, "\mathrm{op}"'] & \mathsf{DeV} \arrow[l, bend left] \arrow[loop below, dashed, "?"]
\end{tikzcd}
$$

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on the category of compact Hausdorff spaces and continuous functions?

Compact Hausdorff space $\leftrightarrow$ Stone space + closed equivalence relation.

$$\begin{tikzcd} \mathsf{Stone} & \ni & \mathcal{G}(X) \arrow[r, hook] \arrow[d, two heads] & \mathcal{G}(Y) \arrow[d, two heads] & \in & \mathsf{Stone} \\ \mathsf{KHaus} & \ni & X \arrow[r, "f"] & Y & \in & \mathsf{KHaus} \end{tikzcd}$$

Continuous functions $\leftrightarrow$ certain closed relations between covers.

A *closed relation* $R\colon X \nrightarrow Y$ is a subset $R \subseteq X \times Y$ that is closed; equivalently, such that

- $R^{-1}[\text{closed}]$ is closed,
- $R[\text{closed}]$ is closed.

In this same conference...

- *Specification properties in CR-dynamical systems*, Ivan Jelić.
- *Orbit structure in CR-dynamical systems*, Andrew Wood.

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We construct a continuous surjection f on the Cantor fan C such that the dynamical system (C, f) is transitive and the inverse limit of (C, f) is the Lelek fan. In the construction, we use a closed relation on $[0, 1]$.

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We give results about a stronger version of the sensitive dependence on initial condition property of so-called Mahavier dynamical system (X_F^+, σ_F^+), where X_F^+ is the Mahavier product of a closed relation F on a non-empty compact metric space and σ_F^+ is the shift

$$\begin{tikzcd} \mathsf{Stone}^{\mathsf{R}} \arrow[r, bend left] \arrow[r, phantom, "\mathrm{op}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, bend left] \end{tikzcd}$$

closed relations subordinations

Stone spaces	Stone locales	Boolean algebras
Closed relations	**Preframe hom.** (= Scott-cont. funct.)	**Subordinations** (= approximable mapp.)
De Groot self-dual. cl. $\leftrightsquigarrow$ comp. sat.	Lawson self-duality elm. $\leftrightsquigarrow$ Scott-op. filt.	Order-self-duality $\leq \leftrightsquigarrow \geq$

Dual of a closed relation:

$$
\begin{tikzcd}
X \arrow[d, "R"] \\
Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Clop}(X) \\
\mathrm{Clop}(Y) \arrow[u, "S"']
\end{tikzcd}
$$

For $V \in \mathrm{Clop}(Y)$ and $U \in \mathrm{Clop}(X)$:

$$V \mathrel{S} U \iff R^{-1}[V] \subseteq U$$

Example:

$$
\begin{tikzcd}
X \arrow[d, "="] \\
X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Clop}(X) \\
\mathrm{Clop}(X) \arrow[u, "\subseteq"']
\end{tikzcd}
$$

Subordination := a relation $S\colon A \looparrowright B$ such that

$$\left(\bigvee_{i=1}^{n} a_i\right) S \left(\bigwedge_{j=1}^{m} b_j\right) \iff \forall i, j \; a_i \; S \; b_j.$$

Theorem (Celani, 2018)

$\mathrm{Stone}^{\mathrm{R}}$ *(closed relations) is dual to* BA^{S} *(subordinations).*

Extension of $\mathbb{V}$ from Stone to $\text{Stone}^{\mathsf{R}}$ [Goy, Petrişan, Aiguier, 2021]:

$$\text{Stone} \xrightarrow{\mathbb{V}^{\mathsf{R}}} \text{Stone}$$

$$\begin{tikzcd} X \arrow[d, hook, "R"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathbb{V}(X) \arrow[d, hook, "\mathbb{V}^{\mathsf{R}}(R)"] \\ \mathbb{V}(Y) \end{tikzcd}$$

(Egli-Milner:) For $K \in \mathbb{V}(X)$ and $L \in \mathbb{V}(Y)$,

$$K \; \mathbb{V}^{\mathsf{R}}(R) \; L \iff \begin{cases} \forall x \in K \; \exists y \in L : \; x \, R \, y, \\ \forall y \in L \; \exists x \in K : \; x \, R \, y. \end{cases}$$

It restricts to the usual Vietoris functor on continuous functions.

$$\begin{tikzcd} \mathsf{Stone}^{\mathsf{R}} \arrow[r, bend left] \arrow[r, phantom, "\mathrm{op}"] \arrow[loop below, "\mathbb{V}^{\mathsf{R}}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, bend left] \arrow[loop below, dashed, "?"] \end{tikzcd}$$

What is the dual of $\mathbb{V}^{\mathsf{R}}$?

On morphisms:

$$\begin{tikzcd}
X \arrow[d, "R"] & \mathbb{V}(X) \arrow[d, "\mathbb{V}^{\mathsf{R}}(R)"] \\
Y & \mathbb{V}(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
B & \mathbb{K}(B) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_b, \Diamond_b \mid b \in B\})/{\sim} \\
A \arrow[u, "S"] & \mathbb{K}(A) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})/{\sim} \arrow[u, "\mathbb{K}^{\mathsf{S}}(S)?"]
\end{tikzcd}$$

We shall describe when an element α of $\mathbb{K}(A)$ is $\mathbb{K}^{\mathsf{S}}(S)$-related with an element β of $\mathbb{K}(B)$.

Let X be a Stone space, and A, B, C, D clopens. Solve:

$$(\Diamond A \cup \Box C) \cap \Box B \subseteq \Diamond C \cup \Box D.$$

$$(\Diamond A \cap \Box B) \cup (\Box C \cap \Box B) \subseteq \Diamond C \cup \Box D.$$

$$\Diamond A \cap \Box B \subseteq \Diamond C \cup \Box D.$$

$$\Diamond (A \cap B) \cap \Box B \subseteq \Diamond C \cup \Box (C \cup D).$$

$$(A \cap B \subseteq C) \text{ or } (B \subseteq C \cup D).$$

$$\Box C \cap \Box B \subseteq \Diamond C \cup \Box D.$$

$$\vdots$$

(Always)

Key idea: $\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$ [Cederquist, Coquand, 1998]

Let X be a Stone space, let $A_1, \dots, A_n, B, C, D_1, \dots, D_m$ be clopens with $A_i \subseteq B$ and $C \subseteq D_j$:

$$\Diamond A_1 \cap \cdots \cap \Diamond A_n \cap \Box B \subseteq \Diamond C \cup \Box D_1 \cup \cdots \cup \Box D_m$$

$$\Updownarrow$$

$$(\exists i : A_i \subseteq C) \text{ or } (\exists j : B \subseteq D_j).$$

Theorem (A., Bezhanishvili, Carai, 2024)

The dual of the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ *is the following endofunctor* $\mathbb{K}^{\mathsf{S}} \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$*:*

- *On objects: it maps* A *to*

$$\mathbb{K}(A) := \frac{\mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\textit{modal algebra axioms}}$$

- *On morphisms: it maps a subordination* $S \colon A \looparrowright B$ *to the unique subordination* $\mathbb{K}^{\mathsf{S}}(S) \colon \mathbb{K}(A) \looparrowright \mathbb{K}(B)$ *satisfying "*$\Diamond$*-with-*$\Diamond$ *or* $\Box$*-with-*$\Box$*".*

"$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$": (With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \; \mathbb{K}^{\mathsf{S}}(S) \; (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \; S \; c) \text{ or } (\exists j : b \; S \; d_j).$$

An application

We apply it to solve

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on compact Hausdorff spaces?

Conclusions

Key ideas

1. Beyond functions: closed relations $\leftrightarrow$ preframe hom. $\leftrightarrow$ subordinations between Boolean algebras.
2. For the Vietoris's dual: "$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$" [Cederquist, Coquand, 1998] (see also [Kawai, 2020]):

$$\left(\bigwedge_i \Diamond_{a_i}\right) \wedge \Box_b \leq \Diamond_c \vee \left(\bigvee_j \Box_{d_j}\right) \Leftrightarrow (\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

3. Our packaging of these ideas:
   - Stone dual description of $\mathbb{V}^{\mathrm{R}}\colon \mathsf{Stone}^{\mathrm{R}} \to \mathsf{Stone}^{\mathrm{R}}$;
   - de Vries dual description of $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ and for relations.

M. Abbadini, G. Bezhanishvili, L. Carai.
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Vietoris has a nice dual.*

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Appendix

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on the category of compact Hausdorff spaces and continuous functions?

De Vries duality connects compact Hausdorff spaces with (Stone spaces and) Boolean algebras.

Every compact Hausdorff space X is a continuous image of a Stone space (e.g., its Gleason cover). So it can be presented via

Stone space + closed equivalence relation.

$$
\begin{array}{ccc}
\mathsf{Stone} & \ni & \mathcal{G}(X) \\
 & & \big\downarrow \\
\mathsf{KHaus} & \ni & X
\end{array}
$$

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$$\begin{tikzcd}
\mathsf{Stone} & \ni & \mathcal{G}(X) \arrow[r, hook] \arrow[d, two heads] & \mathcal{G}(Y) \arrow[d, two heads] & \in & \mathsf{Stone} \\
\mathsf{KHaus} & \ni & X \arrow[r, "f"] & Y & \in & \mathsf{KHaus}
\end{tikzcd}$$

Theorem (A., Bezhanishvili, Carai, 2024)

The de Vries dual of the Vietoris endofunctor on KHaus *is obtained by applying* $\mathbb{K}^{\mathsf{S}}$ *(= the dual of* $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$*), followed by a(n appropriate) MacNeille completion.*

$$X \xrightarrow{R} Y \qquad\qquad (B, \prec_B) \xleftarrow{S} (A, \prec_A)$$

$$\begin{tikzcd}
 & (\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) \arrow[d, hook] & (\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \arrow[l, "\mathbb{K}^{\mathsf{S}}(S)"'] \arrow[d, hook] \\
\mathbb{V}(X) \xrightarrow{\mathbb{V}^{\mathsf{R}}(R)} \mathbb{V}(Y) & \mathbf{M}(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) & \mathbf{M}(\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \arrow[l, "\mathbf{M}(\mathbb{K}^{\mathsf{S}}(S))"']
\end{tikzcd}$$

where $\mathbf{M}$ is an appropriate MacNeille completion functor.