Asymptotic Average Shadowing Versus Vague Specification

Joint work with A. Trilles

Melih Emin Can

Jagiellonian University, Krakow Poland

July 12, 2024

Historical Background

On Shadowing Properties

- The shadowing property was developed independently by Smale, Anasov, and Bowen in mid 20th-century.

Historical Background

On Shadowing Properties

- The shadowing property was developed independently by Smale, Anasov, and Bowen in mid 20th-century.
- Informally, the shadowing property is the possibility of tracing a pseudo-orbit.

Historical Background

On Shadowing Properties

- The shadowing property was developed independently by Smale, Anasov, and Bowen in mid 20th-century.
- Informally, the shadowing property is the possibility of tracing a pseudo-orbit.
- Shadowing property has many (weaker) variants. For example limit shadowing property introduced by Pilyugin et al., average shadowing property introduced by Blank and asymptotic average shadowing property introduced by Gu.

Historical Background

On Specification Properties

- In the early 70's another notion related to tracing a sequence of points in the space by an orbit arose, the specification property, that was introduced by Bowen on the study of Axiom A diffeomorphisms.

Historical Background

On Specification Properties

- In the early 70's another notion related to tracing a sequence of points in the space by an orbit arose, the specification property, that was introduced by Bowen on the study of Axiom A diffeomorphisms.
- In order to study dynamical systems beyond the scope of specification in Bowen's sense, weaker notions of specification were proposed. For example, almost specification, weak specification, relative specification, and others.

Historical Background

Why were shadowing and specification properties studied?

- The both notion are studied because they have many applications to hyperbolic dynamics and topological dynamics.

Historical Background

Why were shadowing and specification properties studied?

- The both notion are studied because they have many applications to hyperbolic dynamics and topological dynamics.
- For topological dynamics the both notion have strong connections to notions of topological entropy, entropy-density of the set of ergodic measures, intrinsically ergodicity of the system and etc.

Motivation

Question [Downarowicz, Weiss (2024)]

Is weak specification property equivalent to vague specification property?

Motivation

Question [Downarowicz, Weiss (2024)]

Is weak specification property equivalent to vague specification property?

Answer

- No, they are not equivalent!

Motivation

Question [Downarowicz, Weiss (2024)]

Is weak specification property equivalent to vague specification property?

Answer

- No, they are not equivalent!
- In fact, we see that weak specification implies vague specification property, but the converse is not true.

Our Strategy

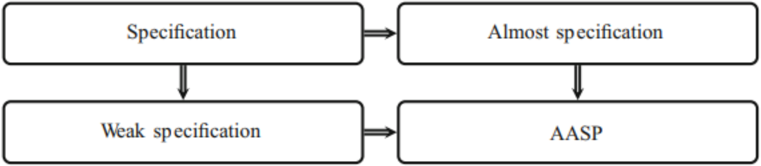

Figure: illustrations of the relations between variants of specification property [1]

[1] This picture taken from the paper [D. Kwietniak, M. Lacka, and P. Oprocha, Generic points for dynamical systems with average shadowing. Monatsh. Math., 183(4):625– 648, 2017]

Basics

- Let (X, T) be a topological dynamical system with a metric ρ on X.

Basics

- Let (X, T) be a topological dynamical system with a metric ρ on X.
- We define **Besicovitch pseudometric** ρ_B on X by

$$\rho_B(x, y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(T(x), T(y)).$$

Basics

- Let (X, T) be a topological dynamical system with a metric ρ on X.
- We define **Besicovitch pseudometric** ρ_B on X by

$$\rho_B(x, y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(T(x), T(y)).$$

- A pair of points (x, y) in a dynamical system $(X \times X, T \times T)$ is called **proximal** if

$$\liminf_{n \to \infty} \rho\left(T^n(x), T^n(y)\right) = 0.$$

Moreover, the system (X, T) is called proximal if every pair in $X \times X$ is proximal.

Basics

- We denote the infinite direct product space of X by X^∞, i.e,

$$X^\infty = \{(x_j)_{j\geq 0} \mid x_j \in X \text{ for all } j \in \mathbb{N}\}.$$

Basics

- We denote the infinite direct product space of X by X^∞, i.e,

$$X^\infty = \{(x_j)_{j\geq 0} \mid x_j \in X \text{ for all } j \in \mathbb{N}\}.$$

- Note that we have a natural shift map S on X^∞ defined by $S((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.

Basics

- We denote the infinite direct product space of X by X^∞, i.e,

$$X^\infty = \{(x_j)_{j\geq 0} \mid x_j \in X \text{ for all } j \in \mathbb{N}\}.$$

- Note that we have a natural shift map S on X^∞ defined by $S((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.
- We denote the sequence $(x_j)_{j\geq 0} \in X^\infty$ by $\underline{x}$.

Basics

- We denote the infinite direct product space of X by X^∞, i.e,

$$X^\infty = \{(x_j)_{j\geq 0} \mid x_j \in X \text{ for all } j \in \mathbb{N}\}.$$

- Note that we have a natural shift map S on X^∞ defined by $S((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.
- We denote the sequence $(x_j)_{j\geq 0} \in X^\infty$ by $\underline{x}$.
- We denote the orbit of x under T by $\underline{x}_T = \{T^n(x)\}_{n=0}^\infty \in X^\infty\}$.

Basics

- We denote the infinite direct product space of X by X^∞, i.e,

$$X^\infty = \{(x_j)_{j\geq 0} \mid x_j \in X \text{ for all } j \in \mathbb{N}\}.$$

- Note that we have a natural shift map S on X^∞ defined by $S((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.
- We denote the sequence $(x_j)_{j\geq 0} \in X^\infty$ by $\underline{x}$.
- We denote the orbit of x under T by $\underline{x}_T = \{T^n(x)\}_{n=0}^\infty \in X^\infty\}$.
- We write $\Gamma_T = \{\underline{x}_T \in X^\infty : x \in X\}$ for the space of all orbits under T.

Pseudo-Orbits

Definition

A sequence $\underline{x} \in X^\infty$ is called an **asymptotic average pseudo-orbit** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(T(x_j), x_{j+1}) = 0.$$

Pseudo-Orbits

Definition

A sequence $\underline{x} \in X^\infty$ is called an **asymptotic average pseudo-orbit** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(T(x_j), x_{j+1}) = 0.$$

Definition

We call $\underline{x} \in X^\infty$ a **vague pseudo-orbit** for T if for any open set $\mathcal{U} \subset X^\infty$ containing Γ_T the following holds:

$$d(\{n \in \mathbb{N} : S^n(\underline{x}) \in \mathcal{U}\}) = 1.$$

Pseudo-Orbits

Theorem 1 [C, A. Trilles]

A sequence $\underline{x} \in X^{\infty}$ is an asymptotic average pseudo-orbit if and only if it is a vague pseudo-orbit.

Tracing Properties

Definition

A system (X, T) has the **asymptotic average shadowing property** if for every asymptotic average pseudo-orbit $\underline{x} \in X^\infty$ for T there exists $z \in X$ such that

$$\rho_B(\underline{z}_T, \underline{x}) = 0.$$

Tracing Properties

Definition

A system (X, T) has the **asymptotic average shadowing property** if for every asymptotic average pseudo-orbit $\underline{x} \in X^\infty$ for T there exists $z \in X$ such that

$$\rho_B(\underline{z}_T, \underline{x}) = 0.$$

Definition

A system (X, T) has the **vague specification property** if for every vague pseudo-orbit $\underline{x} \in X^\infty$ for T there is $z \in X$ such that for every $\varepsilon > 0$ the following holds:

$$d\left(\{n \in \mathbb{N}_0 : \rho(T^n(z), x_n) < \varepsilon\}\right) = 1.$$

Tracing Properties

Theorem 2 [C, A. Trilles]

A system (X, T) has asymptotic average shadowing property if and only if it has vague specification property.

Shift Spaces

- A finite set $\mathscr{A}$ of symbols is called an alphabet. We assume that $\mathscr{A} = \{0, 1\}$. We endow $\mathscr{A}$ with a discrete topology.

Shift Spaces

- A finite set $\mathscr{A}$ of symbols is called an alphabet. We assume that $\mathscr{A} = \{0, 1\}$. We endow $\mathscr{A}$ with a discrete topology.
- The space of infinite sequences of symbols of $\mathscr{A}$ is denoted by $\mathscr{A}^\infty$. That is,

$$\mathscr{A}^\infty = \{(x_j)_{j \geq 0} \mid x_j \in \mathscr{A}\}.$$

Shift Spaces

- A finite set $\mathscr{A}$ of symbols is called an alphabet. We assume that $\mathscr{A} = \{0, 1\}$. We endow $\mathscr{A}$ with a discrete topology.
- The space of infinite sequences of symbols of $\mathscr{A}$ is denoted by $\mathscr{A}^\infty$. That is,

$$\mathscr{A}^\infty = \{(x_j)_{j \geq 0} \mid x_j \in \mathscr{A}\}.$$

- Note that we have a shift map σ on $\mathscr{A}^\infty$ is given by $\sigma((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.

Shift Spaces

- A finite set $\mathscr{A}$ of symbols is called an alphabet. We assume that $\mathscr{A} = \{0, 1\}$. We endow $\mathscr{A}$ with a discrete topology.
- The space of infinite sequences of symbols of $\mathscr{A}$ is denoted by $\mathscr{A}^\infty$. That is,

$$\mathscr{A}^\infty = \{(x_j)_{j\geq 0} \mid x_j \in \mathscr{A}\}.$$

- Note that we have a shift map σ on $\mathscr{A}^\infty$ is given by $\sigma((x_j)_{j\geq 0}) = (x_{j+1})_{j\geq 0}$.
- A set $X \subset \mathscr{A}^\infty$ is called shift space if it is non-empty, closed and invariant.

Shift Spaces

- We call a finite sequence of letters a word over $\mathscr{A}$. The number of letters in a word w is called the length of w, and we denote the length by $|w|$. Let $\mathscr{A}^*$ denotes the set of all words over $\mathscr{A}$.

Shift Spaces

- We call a finite sequence of letters a word over $\mathscr{A}$. The number of letters in a word w is called the length of w, and we denote the length by $|w|$. Let $\mathscr{A}^*$ denotes the set of all words over $\mathscr{A}$.
- We write $x_{[i,j)}$ for the word $x_i x_{i+1} \ldots x_{j-1}$ over $\mathscr{A}$.

Shift Spaces

- We call a finite sequence of letters a word over $\mathscr{A}$. The number of letters in a word w is called the length of w, and we denote the length by $|w|$. Let $\mathscr{A}^*$ denotes the set of all words over $\mathscr{A}$.
- We write $x_{[i,j)}$ for the word $x_i x_{i+1} \dots x_{j-1}$ over $\mathscr{A}$.
- The language of a shift space X is denoted by $\mathcal{L}(X)$ and defined as

$$\mathcal{L}(X) = \left\{ w \in \mathscr{A}^* \mid w = x_{[i,j)} \text{ for some } x \in X \right\}.$$

Shift Spaces

- We define a pseudometric on $\mathscr{A}^\infty$ by

$$\bar{d}(x,y) = \limsup_{n\to\infty} \frac{1}{n} \left|\{0 \le j < n \mid x_j \ne y_j\}\right|,$$

for $x = (x_j)_{j\ge 0} \in \mathscr{A}^\infty$, $y = (y_j)_{j\ge 0} \in \mathscr{A}^\infty$.

- We define a pseudometric on $\mathscr{A}^\infty$ by

$$\bar{d}(x,y) = \limsup_{n\to\infty} \frac{1}{n} \left|\{0 \le j < n \mid x_j \ne y_j\}\right|,$$

for $x = (x_j)_{j\ge 0} \in \mathscr{A}^\infty$, $y = (y_j)_{j\ge 0} \in \mathscr{A}^\infty$.

Definition

We say that a shift space X has the $\bar{d}$-shadowing property if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every sequence $\{w^{(i)}\}_{i=1}^{\infty} \subset \mathcal{L}(X)$ with $|w^{(i)}| \ge N$ for every $i \in \mathbb{N}$ there exists $x \in X$ such that

$$\bar{d}(x,w) < \varepsilon,$$

where $w = w^{(1)}w^{(2)}\dots$.

Theorem 3 [C, A. Trilles]

Assume that (X, σ) is surjective. Then X has vague specification property if and only if X is $\bar{d}$-complete, and has $\bar{d}$-shadowing property.

Example of Proximal Shift Spaces with VSP

Example

Let $n \in \mathbb{N}$. We construct a sofic shifts Z_n represented by an oriented labeled graph $G_n = (V_n, E_n, \tau_n)$ with vertex set $V_n = \{v_0, v_1, \ldots, v_{10^n-1}\}$ and edge set E_n and labels given by:

- for every $0 \le k < 10^n$, there is an edge from v_k to v_{k+1} with label 0, where $v_{10^n} = v_0$;

Example of Proximal Shift Spaces with VSP

Example

Let $n \in \mathbb{N}$. We construct a sofic shifts Z_n represented by an oriented labeled graph $G_n = (V_n, E_n, \tau_n)$ with vertex set $V_n = \{v_0, v_1, \ldots, v_{10^n-1}\}$ and edge set E_n and labels given by:

- for every $0 \leq k < 10^n$, there is an edge from v_k to v_{k+1} with label 0, where $v_{10^n} = v_0$;
- for every $1 \leq k \leq 10^n - 2^n$, there is an edge from v_k to v_{k+1} with label 1,

Example of Proximal Shift Spaces with VSP

Example

Let $n \in \mathbb{N}$. We construct a sofic shifts Z_n represented by an oriented labeled graph $G_n = (V_n, E_n, \tau_n)$ with vertex set $V_n = \{v_0, v_1, \ldots, v_{10^n-1}\}$ and edge set E_n and labels given by:

- for every $0 \leq k < 10^n$, there is an edge from v_k to v_{k+1} with label 0, where $v_{10^n} = v_0$;
- for every $1 \leq k \leq 10^n - 2^n$, there is an edge from v_k to v_{k+1} with label 1,
- there is an edge from $v_{10^n-2^n}$ to $v_{10^n-2^n+2}$ with label 0.

Example of Proximal Shift Spaces with VSP

Example

Let $n \in \mathbb{N}$. We construct a sofic shifts Z_n represented by an oriented labeled graph $G_n = (V_n, E_n, \tau_n)$ with vertex set $V_n = \{v_0, v_1, \ldots, v_{10^n - 1}\}$ and edge set E_n and labels given by:

- for every $0 \leq k < 10^n$, there is an edge from v_k to v_{k+1} with label 0, where $v_{10^n} = v_0$;
- for every $1 \leq k \leq 10^n - 2^n$, there is an edge from v_k to v_{k+1} with label 1,
- there is an edge from $v_{10^n - 2^n}$ to $v_{10^n - 2^n + 2}$ with label 0.

We set $Z = \bigcap_{n=1}^{\infty} Z_n$.

Example of Proximal Shift Spaces with VSP

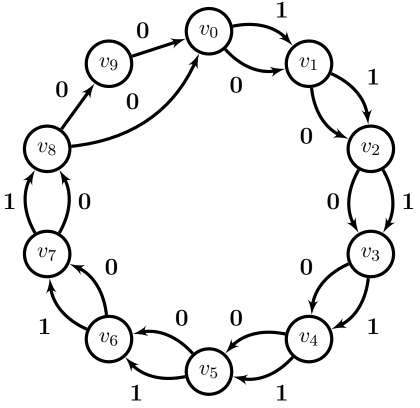

Figure: The graph of the Example with $n = 1$.

Example of Proximal Shift Spaces with VSP

Theorem [C, Konieczny, Kupsa, Kwietniak]

The shift space Z in the example is surjective, mixing, proximal, hereditary, has positive entropy, and has $\bar{d}$-shadowing property.

Example of Proximal Shift Spaces with VSP

Theorem [C, Konieczny, Kupsa, Kwietniak]

The shift space Z in the example is surjective, mixing, proximal, hereditary, has positive entropy, and has $\bar{d}$-shadowing property.

Theorem 4 [C, A. Trilles]

The shift space Z in the example is $\bar{d}$-complete. Hence, it has vague specification property.